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THE S-ASYMPTOTIC AND OTHER DEFINITIONS OF THE ASYMPTOTIC BEHAVIOUR OF DISTRIBUTIONS

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ABSTRACT

The paper compares different definitions of the behaviour of a distribution at infinity with the S-asymptotic by some examples and propositions.

INTRODUCTION

In the last thirty years, many definitions of the asymptotic behaviour of distributions have been presented. We can divide them into two sets. To the first one belong those definitions which directly use the classical definition of the asymptotic behaviour of a numerical function. All of them are given only in the one dimensional case. The second set contains definitions which correspond to the distribution a class of distributions depending on some parameters.

Representatives of the first set are definitions given by M.J. Lighthill [5] and by J. Lavoine and O.P. Misra [4].

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Key words and phrases: Asymptotic behaviour of distributions, quasiasymptotic, equivalence at infinity, S-asymptotic.

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Definition 1. (Lighthill's definition). *Distribution* T *behaves as the numerical function* f(x) *at a point* x_0 *if and only if* T *is equal to a numerical function* F *in an open neighbourhood of* x_0 *and* $F(x) \sim f(x)$, $x \to x_0$.

The definition used by J. Lavoine and O.P. Misra has been improved by other mathematicians. We shall cite the last we know [8].

Definition 2. (A. Takači's definition). *The distribution* T *is equivalent at infinity with a regularly varying function* $\rho(x) = x^p L(x)$, $p \neq -n$, $n \in N$, *if there exist* $n \in N_0$, $n+p>0$, $a > 1$ *and a continuous function* F *on* R *such that* $T = D^n F(x)$ *on* (a,∞) *and* $F(x) \sim C_{p,n} x^{p+n} L(x)$, $x \to \infty$. *We write then* $T \overset{E}{\sim} \rho(x)$, $x \to \infty$.

L(x) is a slowly varying function, and the constant $C_{p,n}$

$$C_{p,n} = \begin{cases} 1/(p+1) \dots (p+n), & n \in N \\ 1, & n = 0. \end{cases}$$

Representatives of the second set are: the quasiasymptotic and S-asymptotic.

The most general definition of the quasiasymptotic is given in book [9]. Before we give this definition, we have to introduce some notations.

Let Γ be a closed, convex, acute and bodied cone in R^n;

$\{U_k, k \in I\}$ be a set of linear applications defined in R^n such that $U_k \Gamma = \Gamma$, $k \in I$ and *det* $U_k > 0$, $k \in I$;

$\rho(k)$ be a positive function, $k \in I$;

I be a subset of R having ∞ as a limit point.

Definition 3. (Quasiasymptotic). *Suppose that* $T \in S'$ *with its support in* Γ. T *has the quasiasymptotic in* Γ *related to the set* $\{U_k, k \in I\}$ *and the function* $\rho(k)$ *if in* (S')

$$\frac{1}{\rho(k)} T(U_k t) \to g(t), \quad k \to \infty, \quad k \in I.$$

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In our examples we shall use only the one-dimensional case with $\Gamma = [0,\infty)$.

Definition 4. (S-asymptotic). *A distribution* $T \in (D')$ *has the S-asymptotic in cone* Γ, *related to some* $c(h) > 0$, $h \in \Gamma$, *and with the limit* $U \in (D')$ *if there exists*

$$\lim_{h\in\Gamma,\|h\|\to\infty} < T(t+h)/c(h),\varphi(t) > = < U,\varphi > , \quad \varphi \in (D).$$

Then we write $T(t+h) \overset{S}{\sim} c(h)U(t)$, $h \in \Gamma$, [7].

1. S-ASYMPTOTIC OF A REGULAR DISTRIBUTION AND ASYMPTOTIC BEHAVIOUR OF FUNCTIONS AT INFINITY

According to Lighthill's definition, we have cited, of the behaviour of a distribution at infinity and bearing in mind the local property of the S-asymptotic [7], it is enough to compare the S-asymptotic and asymptotic of numerical functions and we have at the same time a comparison of the S-asymptotic and Lighthill's definition of the behaviour of a distribution at infinity.

The following example (see [1], p. 45) points out that a continuous function can have the S-asymptotic as a distribution without having the asymptotic: Let

$$T(\tau) = \int_{\alpha}^{\tau} g(x)dx, \quad g \in L^1(-\infty,\infty) \cap C(-\infty,\infty), \quad \alpha > 0.$$

Then

$$T(\tau+\beta) \overset{S}{\sim} 1 \cdot \int_{\alpha}^{\infty} g(x)dx, \quad \beta \in R_+.$$

By Theorem 1 c) [7] $T^{(1)}(t+\beta) \overset{S}{\sim} 1 \cdot (\int_{\alpha}^{\infty} g(x)dx)', \beta \in R_+$. Hence $g(t+\beta) \overset{S}{\sim} 1 \cdot 0$, $\beta \in R_+$.

But g must not have the asymptotic behaviour when $\tau \to \infty$. This example shows that every function from $L^1 \cap C$ has an S-asymptotic zero, when $\beta \to \infty$ or $\beta \to -\infty$, related to $c(h) \equiv 1$.

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Let, now, $h(t) = e^t \int_\alpha^t g(x)dx$, where $g(x)$ has the same properties as in the preliminary case. It is easy to see that

$$h(t+\beta) \overset{S}{\sim} e^\beta \cdot e^t \int_\alpha^\infty g(x)dx, \quad \beta \in R_+$$

$h'(t) = h(t) + e^t g(t)$ has the S-asymptotic related to e^β, $\beta \in R_+$, just $e^t \int_\alpha^\infty g(x)dx$. But $h'(t)$ must not have an asymptotic related to e^t because of $g(t)$. All derivatives $h^{(k)}(t)$ have the same S-asymptotic as $h(t)$, related to e^β.

The following example shows that a function can have the usual asymptotic behaviour without having an S-asymptotic with the limit U different from zero. This example will be the function: $exp(x^2)$. Suppose that the regular distribution defined by the function $exp(x^2)$ has the S-asymptotic relative to $c(h)>0$, $h \in R_+$ with a limit U different from zero. By Proposition 4 [7], U has the form $U(t) = C\, exp(at)$. Then for every $\varphi \in (D)$, and consequently for $\varphi > 0$ we have

$$\lim_{h\to\infty} \frac{1}{c(h)} \int exp[(x+h+h_0)^2]\, \varphi(x)dx = e^{ah_0} < Ce^{ax}, \varphi(x) >.$$

Therefore

$$e^{ah_0} < U, \varphi > =$$

$$= exp(h_0^2) \lim_{h\to\infty} \frac{1}{c(h)} \int e^{(x+h)^2} e^{2h_0(x+h)} \varphi(x)dx$$

$$\geq exp(h_0^2) < U, \varphi >, \text{ for every } h_0 > 0.$$

But this is not correct.

It is easy to show that for some classes of numerical functions from the asymptotic behaviour at infinity there follows the S-asymptotic. The next proposition is such a one.

Proposition 1. *Let Γ be a convex cone and $\Omega \subset R^n$*

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an open set with the property: for every $r > 0$ *there exists a* β_r *such that* $B(h,r) \subset \Omega$, $h \in \Gamma$, $\|h\| \geq \beta_r$.

Suppose that the function G *is local integrable over* Ω *and has the following properties:*

a) The distribution G_0 *is equal on* Ω *to the distribution* $\tilde{G}_\Omega$ *defined by the function* G.

b) There exist number M_r *and a local integrable function* V *such that*

$$|G(x+h)/c(h)| \leq M_r, \quad x \in B(0,r), \quad x+h \subset \Omega ;$$

$$\lim_{h\in\Gamma, \|h\|\to\infty} G(x+h)/c(h) = V(x), \quad x \in B(0,r).$$

Then $G_0(x+h) \overset{S}{\sim} c(h)V(x)$, $h \in \Gamma$.

Proof. Because of the local property of the S--asymptotic [7] it is enough to prove that for a $\varphi \in D$, $supp\varphi \subset B(0,r)$

$$\lim_{h\in\Gamma, \|h\|\to\infty} \int \frac{G(x+h)}{c(h)} \varphi(x)dx = \int V(x)\varphi(x)dx.$$

Bearing in mind the property of Ω and supposition b), we can go to the limit in h under the integral sign.

2. RELATION BETWEEN THE QUASIASYMPTOTIC AND S-ASYPTOTIC

To compare these two notions we have first to fix the basic space and the class of numerical functions c(h). As the quasiasymptotic is defined for tempered distributions (S') and related to regular varying functions ($c(h) = h^a L(h)$, $a \in R$ and L(h) is a slowly varying function) we shall compare, first, in this case, supposing moreover that the limit distribution U differs from zero.

The following example shows the imperfection of the comparison of these two notions. In the following $\theta(t) = 1$, $t > 0$; $\theta(t) = 0$, $t < 0$

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i) The regular distribution $\tilde{T} = \theta(t)\, e^{iat}$, $a \neq 0$, has the quasiasymptotic $\frac{i}{a}\delta$ in (S') related to $c(h) = h^{-1}$ [2]:

$$k < \theta(kt)\, e^{ikat}, \varphi(t) > = k \int_0^\infty e^{ikat}\varphi(t)dt$$

$$= \frac{1}{ia} \int_0^\infty \varphi(\frac{x}{k}) d(e^{iax}) =$$

$$= \frac{1}{ia}\, e^{iax}\varphi(\frac{x}{k})\Big|_0^\infty - \frac{1}{k} \int_0^\infty e^{iax}\varphi'(\frac{x}{k})dx$$

$$\to \frac{i}{a}\varphi(0), \quad k \to \infty.$$

But the distribution $\tilde{T}$ has no S-asymptotic related to a h^α with a $U \neq 0$:

$$< \theta(t+h)e^{ia(t+h)}, \varphi(t) > = e^{iah} \int_{-h}^\infty e^{iat}\varphi(t)dt$$

$$\sim e^{iah} \int_{-\infty}^\infty e^{iat}\varphi(t)dt, \quad h \to \infty.$$

This distribution really has an S-asymptotic but relative to the function $c(h) = e^{iah}$.

ii) The regular distribution $T(t) = \theta(t)\sin t$ has quasiasymptotic relative to $c(h) = h^{-1}$ but it has no S-asymptotic at all:

$$h < \theta(ht)\sin ht, \varphi(t) > = \int_0^\infty \sin u\varphi(\frac{u}{h})du$$

$$= \varphi(0) + \frac{1}{h} \int_0^\infty \varphi'(\frac{u}{h})\cos u\, du.$$

For the S-asymptotic we have

$$< \theta(t+h)\sin(t+h), \varphi(t) > \sim \cos h \int_{-\infty}^\infty \sin t\varphi(t)dt +$$

$$+ \sin h \int_{-\infty}^\infty \cos t\varphi(t)dt, \quad h \to \infty.$$

iii) For the regular distribution $\tilde{T} = \theta(t)\sin\sqrt{t}$ we can not find an $\alpha \in R$ and a distribution $U_\alpha \neq 0$ such that

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$$\lim_{k\to\infty} \int_0^\infty k^\alpha \sin\sqrt{kt}\ \varphi(t)dt = \langle U_\alpha, \varphi \rangle, \quad \varphi \in (S).$$

Suppose on the contrary that such α and U_α exist. Then we can choose for the φ, $\varphi(t) = e^{-pt}$, $t > 0$, $\mathrm{Re}\ p > 0$. Then we have

$$\lim_{k\to\infty} k^\alpha \int_0^\infty \sin\sqrt{kt}\ e^{-pt}dt = \langle U_\alpha(t), e^{-pt} \rangle.$$

The value of the last integral is $\sqrt{\pi k}/\sqrt{4p^3}\ \exp(-k/4p)$ and $\langle U_\alpha(t), e^{-pt} \rangle$ is the Laplace transform of distribution U_α. In such a way the last relation says that the Laplace transform of U_α equals zero for $\mathrm{Re}\ p > 0$, hence $U_\alpha = 0$.

A proposition which compares these two notions is the following [2]:

Proposition 2. *Suppose that* $f \in (S'_+(R))$ *and* $f(t+h) \overset{S}{\sim} h^\alpha U(t)$, $h \to \infty$, $\alpha > -1$. *Then f has a quasiasymptotic of order* α *as well.*

Proof. By relation (6) $\langle f(x+h), \varphi(x) \rangle = (f * \check{\varphi})(h) = H(h)$. This numerical function $H(h)$ has the usual asymptotic of order α, when $h \to \infty$. By Lemma 1 [2], it has the quasiasymptotic at infinity of the same order α. The Fourier transform of distribution f gives $F[f(kx)](p) = \frac{1}{k} F[f](\frac{p}{k})$. From the continuity of the Fourier transform follows that $F[f]$ has the quasiasymptotic at zero of order $\alpha+1$ if and only if f has the quasiasymptotic at infinity of order α.

Suppose now that we chose an $\varphi = \varphi_0 \in (S)$ such that $F[\check{\varphi}](p) = 1$ in a neighbourhood of zero. The Fourier transform of $H(h) = (f * \check{\varphi})(h)$ gives $F[H] = F[f]F[\check{\varphi}]$. We know that $F[H](p)$ has the quasiasymptotic at zero of order $\alpha+1$. With the φ_0 we chose, there follows that $F[f]$ has the quasiasymptotic at zero of order $\alpha+1$ as well. Therefore f has the quasiasymptotic at infinity of the order α.

If we change the basic space, the conclusion can be quite different. Suppose that the basic space is (K_1) (see [6])

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Elements of (K_1) are functions φ from C^∞ such that

$$\nu_k(\varphi) = \sup_{x\in R, n<k} e^{k|x|}|D^n\varphi| < \infty .$$

The function $\theta(x)\,Chx = \frac{1}{2}\theta(x)(e^x + e^{-x})$ defines an element from (K_1') and $exp(-px^2)$, $p > 0$, is from (K_1).

To find the quasiasymptotic at infinity in (K_1') of $\theta(x)Ch(x)$ we shall analyse the integral

$$\int_0^\infty Ch(kx)e^{-px^2}dx = \frac{1}{2}\int_0^\infty Ch(k\sqrt{x})e^{-px}\frac{dx}{\sqrt{x}} = \frac{\sqrt{\pi}}{\sqrt{4p}}exp(k^2/4p).$$

This shows that there exists no function $c(k)$ of the form $e^{ak^r}k^bL(k)$, $a,b \in R$, $0 \le r \le 2$, $L(k)$ slowly varying function, such that $\theta(x)Chx$ has the quasiasymptotic relative to $c(k)$ with a limit $U \neq 0$.

For the S-asymptotic we have for $\varphi \in (K_1)$:

$$\lim_{h\to\infty} e^{-h}\int_{-h}^{\infty} C(x+h)\varphi(x)dx =$$

$$= \frac{1}{2}\int_{-h}^{\infty} e^x\varphi(x)dx + \frac{1}{2}e^{-2h}\int_{-h}^{\infty} e^{-x}\varphi(x)dx$$

$$\to \frac{1}{2}\int_{-\infty}^{\infty} e^x\varphi(x)dx, \quad h \to \infty.$$

Therefore $\theta(x)Chx$ has the S-asymptotic relative to $c(h) = e^h$ with the limit $U = \frac{1}{2}e^x$.

3. EQUIVALENCE AT INFINITY OF A DISTRIBUTION AND THE S-ASYMPTOTIC

By Definition 2, the equivalence at infinity of a distribution is related only to the power growth such as $x^pL(x)$ of a distribution. Therefore, the comparison has a meaning only for such distributions.

If we have regular distributions, the difference between these two definitions of asymptotic behaviour of dis-

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tributions follows from the fact that the S-asymptotic preserves the usual asymptotic of numerical functions (Theorem 1.b) [7]), but the equivalence at infinity generalizes l'Hospital's rule with Stolz's improvement. Namely, this rule says: Suppose that the real valued function $H(x)$ has the first derivative for $x > x_0$; $H'(x) \neq 0$, $x > x_0$ and $H(x) \to \infty$, $x \to \infty$; the function $F(x)$ has its first derivative. If there exists

$$\lim_{x\to\infty} F'(x)/H'(x) = A, \text{ then there exists}$$

$$\lim_{x\to\infty} F(x)/H(x) = A.$$

We know that the opposite does not hold. Definition 2 uses just the opposite statement in l'Hospital's rule, with $H(x) = x^{p+1}$. The next example shows this disagreement with the usual asymptotic by numerical functions:

$$F(x) = \frac{1}{2} x^{-\frac{1}{2}} + e^x \cos e^x = D(x^{\frac{1}{2}} + \sin e^x).$$

By Definition 2, the distribution defined by function $F(x)$ is equivalent at infinity with $\frac{1}{2} x^{-\frac{1}{2}}$.

By reason of the same fact, the function

$$F(x) = x^{-\frac{1}{2}} + \sin x = D(2x^{\frac{1}{2}} - \cos x)$$

is equivalent at infinity with $x^{-\frac{1}{2}}$ but has no S-asymptotic with a limit $U \neq 0$ (see 2 . ii)).

Without any difficulty one can find a function which has the S-asymptotic but for which Definition 2 does not hold. We shall give such a one, not trivial:

$$F(x) = e^{-x} \sin(\frac{\pi}{2} - e^{-x}) = D(\sin(\frac{\pi}{2} - e^{-x})).$$

For $x > 0$

$$1 \geq \sin(\frac{\pi}{2} - e^{-x}) \geq \sin(\frac{\pi}{2} - 1).$$

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Hence

$$\frac{x^{n-1}}{(n-1)!} \geq \int_0^x \int_0^y \dots \int_0^t \sin\left(\frac{\pi}{2} - e^{-u}\right) du \dots dy$$

$$\geq \frac{x^{n-1}}{(n-1)!} \sin\left(\frac{\pi}{2} - 1\right), \quad n \in N.$$

The function $F(x)$ can be written in the form $F(x) = D^n E(x)$, where $E(x)$ is

$$E(x) = \int_0^x \int_0^y \dots \int_0^t \sin\left(\frac{\pi}{2} - e^{-u}\right) du \dots dy.$$

From the previous inequality it follows that the behaviour at infinity could be x^{n-1}, but Definition 2 does not admit such a power.

Proposition 3. *Suppose that* T *is equivalent at infinity with* $Ax^p L(x)$, *then there exists* n_0 *such that* T *has the S-asymptotic related to* $c(h) = h^{p+n} L(h)$, $n \geq n_0$ *and with the limit* $U = 0$.

Proof. By Definition 2, there exists a continuous function $f(x)$, $x \in R$ such that $T = D^k f(x)$, $x > x_0$ and

$$f(x) \sim Ax^{p+k}/(p+1) \dots (p+k) L_1(x), \quad x \to \infty.$$

Function $f(x)$ has the property that for a $C \in R^+$ and every compact set $K \subset R$

$$\left|\frac{f(x+h)}{h^{p+k} L(h)}\right| = \left|\frac{f(x+h)}{(x+h)^{p+k} L(h)}\right| \left|\frac{(x+h)^{p+k}}{h^{p+k}}\right| \leq C, \quad x \in K, \; h \geq h_K$$

and

$$\lim_{h \to \infty} f(x+h)/h^{p+k} L(h) = A/(p+1) \dots (p+k).$$

By Theorem 1.b) [7] $f(x+h) \overset{S}{\sim} h^{p+k} L(h) A/(p+1) \dots (p+k)$, $h \to \infty$,

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$h > 0$. Now from Theorem 1.c) [7] follows the statement of our proposition with $n_0 = k$.

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COINCIDENCE POINTS FOR SET-VALUED MAPPINGS IN CONVEX METRIC SPACES

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ABSTRACT

In this paper a generalization of the well-known fixed point theorem of Assad and Kirk for multivalued mappings in convex metric spaces is given. The multivalued version of the Palais-Smale condition is introduced and applied in the proof of Theorem 2., which contains an existence result on coincidence points for set-valued mappings in metric spaces with a convex structure.

1. INTRODUCTION

Since Assad and Kirk published their paper [1] many authors proved fixed point theorems or theorems on coincidence points in metric spaces with a convex structure or in convex metric spaces [2], [5], [6], [7], [8], [9], [10], [11], [12].

Let us recall that a metric space (M,d) is convex if for each $x,y \in M$ with $x \neq y$ there exists $z \in M$, $x \neq z \neq y$ such that

$$d(x,z) + d(z,y) = d(x,y).$$

In [1] the following result is obtained.

THEOREM A. *Let* (M,d) *be a complete convex metric space,* K *a nonempty closed subset of* M, $f: K \to CB(M)$ *(the family of all nonempty closed and bounded subsets of* M*) so that* $f(\partial K) \subseteq K$ *and* f *is a contraction mapping (in respect to the Hausdorff metric* H*). Then there exists*

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$z \in K$ *such that* $z \in fz$.

Itoh [7], Khan [8], Baskaran and Subrahmanyam [2] and Hadžić [6] (for singlevalued f) obtained generalizations of Theorem A. Theorem 1, which will be proved in this paper, is a theorem on the coincidence point for mappings $f: K \to CB(M)$ $S: K \to M$ and $T: K \to M$. If $S = T = Id|_K$ from Theorem 1 it follows Theorem A. A point $y \in K$ is a coincidence point for f, S and T if

$$\{Ty, Sy\} \subseteq fy .$$

We shall need for the next text some notions and notations. If X is a metric space, we shall denote by 2^X the family of all nonempty subsets of X. Let (X,d) be a metric space, $f: K \to CB(X)$, $S: K \to X$ and $\emptyset \neq K \subset X$. The pair (f,S) is said to be weakly commutative if and only if for every $y \in K$ and $z \in K$ such that $y \in fz$ and $Sz \in K$:

$$d(Sy, fSz) \leqslant d(fz, Sz) .$$

For singlevalued mappings the notion of the weak commutativity is introduced by Sessa in [11]. There are examples of mappings which are weakly commutative but not commutative. If f and S are such that for $Sz \in K$ and $fz \subset K$, $fSz = Sfz$ then the pair (f,S) is obviously weakly commutative since for $y \in fz \cap K$ and $Sz \in K$:

$$d(Sy, fSz) = \inf_{u \in fSz} d(Sy, u)$$

and there exists $v \in fSz$ such that $Sy = v$. This implies that

$$\inf_{u \in fSz} d(Sy, u) \leqslant d(Sy, v) = d(v, v) = 0 \leqslant d(fz, Sz)$$

In [12] Takahashi introduced the notion of the convexity in metric spaces. Let (X,d) be a metric space and $W: X \times X \times [0,1] \to X$. The mapping W is said to be a convex structure if for every $(x,y,\lambda) \in X \times X \times [0,1]$:

$$d(u, W(x,y,\lambda)) \leqslant \lambda d(u,x) + (1-\lambda) d(u,y)$$

for every $u \in X$. A metric space with a convex structure belongs to the class of convex metric spaces. There are metric spaces with a convex structure which can not be imbedded in

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any Banach space [12].

A metric space (X,d) with a convex structure W satisfies condition II if for all $(x,y,z,\lambda) \in X^3 \times [0,1]$

$$d(W(x,z,\lambda),W(y,z,\lambda)) \leqslant \lambda d(x,y) \quad [10].$$

If (X,d) is a metric space with a convex structure $W, x_0 \in X$ and $S:X \to X$ we say that the mapping S is (W,x_0)-convex if and only if for every $z \in X$ and every $\lambda \in (0,1)$:

$$W(Sz,x_0,\lambda) = SW(z,x_0,\lambda) .$$

By α we shall denote the Kuratowski measure of noncompactness.

2. A GENERALIZATION OF THE FIXED POINT THEOREM OF ASSAD AND KIRK

THEOREM 1. *Let* (M,d) *be a complete convex metric space,* K *a nonempty closed subset of* M, $S,T:K \to M$ *continuous mappings,* $f:K \to CB(M)$ *H-continuous mapping,* $\partial K \subseteq SK \cap TK$, $fK \cap K \subseteq SK \cap TK$, (f,S) *and* (f,T) *weakly commutative pairs and the following implications hold:*

$$Tx \in \partial K \Rightarrow fx \in K ; \quad Sx \in \partial K \Rightarrow fx \in K .$$

If there exists $q \in (0,1)$ *so that:*

$$H(fx,fy) \leqslant qd(Sx,Ty) , \text{ for every } x,y \in K ,$$

then there exists $z \in K$ *so that:*

$$\{Tz,Sz\} \cap fz \neq \emptyset .$$

If $S,T:M \to M$ *are continuous and:*

(i) $\quad y \in fx,\ Tx \in K \Rightarrow d(Ty,fTx) \leqslant d(fx,Tx)$

(ii) $\quad y \in fx,\ Sx \in K \Rightarrow d(Sy,fSx) \leqslant d(fx,Sx)$

then there exists $z \in K$ *so that:*

$$Tz \in fz \text{ and } Sz \in fz .$$

PROOF : Let $x \in \partial K$. Since $\partial K \subseteq TK$ it follows that there exists $p_0 \in K$ such that $x = Tp_0$.

From $Tp_0 \in \partial K$, using the implication: $Tx \in \partial K \Rightarrow fx \in K$, we conclude that $fp_0 \in K \cap fK \subseteq SK$. Let $p_1 \in K$ be such that $Sp_1 = p_1' \in fp_0 \subseteq K$. Since $p_1' \in fp_0$ there exists $p_2' \in fp_1$ so that $d(p_1',p_2') \leqslant H(fp_0,fp_1) + q$. Suppose that $p_2' \in K$. Then

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$p_2' \in K \cap f(K) \subseteq TK$ which implies that there exists $p_2 \in K$ such that $Tp_2 = p_2'$. If $p_2' \notin K$ then there exists $q \in \partial K$ so that:

$$d(Sp_1, q) + d(q, p_2') = d(Sp_1, p_2') .$$

Since $q \in \partial K \subseteq TK$ there exists $p_2 \in K$ such that $q = Tp_2$ and so:

$$d(Sp_1, Tp_2) + d(Tp_2, p_2') = d(Sp_1, p_2') .$$

Let $p_3' \in fp_2$ be such that:

$$d(p_2', p_3') \leqslant H(fp_1, fp_2) + q^2 .$$

It is easy to see that in this way we obtain two sequences $\{p_n\}$, $n \in N$ and $\{p_n'\}$, $n \in N$ such that

1. For every $n \in N$: $p_n' \in fp_{n-1}$

2. For every $n \in N$: $p_{2n}' \in K \Rightarrow p_{2n}' = Tp_{2n}$
$p_{2n}' \notin K \Rightarrow Tp_{2n} \in \partial K$ and

(1) $$d(Sp_{2n-1}, Tp_{2n}) + d(Tp_{2n}, p_{2n}') = d(Sp_{2n-1}, p_{2n}') .$$

3. For every $n \in N$: $p_{2n+1}' \in K \Rightarrow p_{2n+1}' = Sp_{2n+1}$
$p_{2n+1}' \notin K \Rightarrow Sp_{2n+1} \in \partial K$ and :

(2) $$d(Tp_{2n}, Sp_{2n+1}) + d(Sp_{2n+1}, p_{2n+1}') = d(Tp_{2n}, p_{2n+1}') .$$

4. For every $n \in N$:

$$d(p_n', p_{n+1}') \leqslant H(fp_{n-1}, fp_n) + q^n .$$

Let P_o, P_1, Q_o and Q_1 be defined by:

$$P_o = \{p_{2n},\ n \in N \text{ and } p_{2n}' = Tp_{2n}\}$$

$$P_1 = \{p_{2n},\ n \in N \text{ and } p_{2n}' \neq Tp_{2n}\}$$

$$Q_o = \{p_{2n+1},\ n \in N \text{ and } p_{2n+1}' = Sp_{2n+1}\}$$

$$Q_1 = \{p_{2n+1},\ n \in N \text{ and } p_{2n+1}' \neq Sp_{2n+1}\} .$$

First we shall prove that:

$$(p_{2n}, p_{2n+1}) \notin P_1 \times Q_1 \text{ and } (p_{2n-1}, p_{2n}) \notin Q_1 \times P_1 .$$

If $p_{2n} \in P_1$ then $p_{2n}' \neq Tp_{2n}$ and in this case we have that $Tp_{2n} \in \partial K$ which implies that $p_{2n+1}' \in fp_{2n} \subseteq K$. Hence $p_{2n+1}' = Sp_{2n+1}$ which means that $p_{2n+1} \in Q_o$. We can prove similarly that $(p_{2n-1}, p_{2n}) \notin Q_1 \times P_1$.

Let us prove that for every $n \in N$:

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$$(3)\qquad d(Tp_{2n},Sp_{2n+1}) \leqslant \begin{cases} qd(Sp_{2n-1},Tp_{2n}) + q^{2n} \\ \text{or} \\ qd(Sp_{2n-1},Tp_{2n-2}) + q^{2n-1} + q^{2n} \end{cases}$$

and

$$(4)\qquad d(Sp_{2n-1},Tp_{2n}) \leqslant \begin{cases} qd(Sp_{2n-1},Tp_{2n-2}) + q^{2n-1} \\ \text{or} \\ qd(Sp_{2n-3},Tp_{2n-2}) + q^{2n-2} + q^{2n-1} . \end{cases}$$

If $(p_{2n},p_{2n+1}) \in P_o \times Q_o$ then:

$$d(Tp_{2n},Sp_{2n+1}) = d(p'_{2n},p'_{2n+1}) \leqslant H(fp_{2n-1},fp_{2n}) + q^{2n} \leqslant$$

$$\leqslant qd(Sp_{2n-1},Tp_{2n}) + q^{2n} .$$

Suppose that $(p_{2n},p_{2n+1}) \in P_o \times Q_1$. Then from (2) we obtain that:

$$d(Tp_{2n},Sp_{2n+1}) \leqslant d(Tp_{2n},p'_{2n+1}) = d(p'_{2n},p'_{2n+1}) \leqslant$$

$$\leqslant qd(Sp_{2n-1},Tp_{2n}) + q^{2n} .$$

From the relation $(p_{2n},p_{2n+1}) \in P_1 \times Q_o$ we have that:

$$d(Tp_{2n},Sp_{2n+1}) \leqslant d(Tp_{2n},p'_{2n}) + d(p'_{2n},p'_{2n+1}) \leqslant$$

$$\leqslant d(Tp_{2n},p'_{2n}) + qd(Sp_{2n-1},Tp_{2n}) + q^{2n} \leqslant$$

$$\leqslant d(Tp_{2n},p'_{2n}) + d(Sp_{2n-1},Tp_{2n}) + q^{2n} .$$

Using (1) we obtain that:

$$d(Tp_{2n},Sp_{2n+1}) \leqslant d(Sp_{2n-1},p'_{2n}) + q^{2n} .$$

Since $p_{2n} \in P_1$ and $(p_{2n-1},p_{2n}) \notin Q_1 \times P_1$ we obtain that $p_{2n-1} \in Q_o$ which means that $p'_{2n-1} = Sp_{2n-1}$ and so:

$$d(Tp_{2n},Sp_{2n+1}) \leqslant d(p'_{2n-1},p'_{2n}) + q^{2n} \leqslant$$

$$\leqslant qd(Sp_{2n-1},Tp_{2n-2}) + q^{2n-1} + q^{2n} .$$

If $(p_{2n-1},p_{2n}) \in Q_o \times P_o$ then:

$$d(Sp_{2n-1},Tp_{2n}) = d(p'_{2n-1},p'_{2n}) \leqslant qd(Sp_{2n-1},Tp_{2n-2}) + q^{2n-1} .$$

Suppose that $(p_{2n-1},p_{2n}) \in Q_o \times P_1$. Then from (1) we have that:

$$d(Sp_{2n-1},Tp_{2n}) \leqslant d(Sp_{2n-1},p'_{2n}) = d(p'_{2n-1},p'_{2n}) \leqslant$$

$$\leqslant qd(Sp_{2n-1},Tp_{2n-2}) + q^{2n-1} .$$

If $(p_{2n-1},p_{2n}) \in Q_1 \times P_o$ then:

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$$d(Sp_{2n-1},Tp_{2n}) \leqslant d(Sp_{2n-1},p'_{2n-1}) + d(p'_{2n-1},Tp_{2n}) =$$

$$= d(Sp_{2n-1},p'_{2n-1}) + d(p'_{2n-1},p'_{2n}) \leqslant$$

$$\leqslant d(Sp_{2n-1},p'_{2n-1}) + qd(Sp_{2n-1},Tp_{2n-2}) + q^{2n-1} \leqslant$$

$$\leqslant d(Sp_{2n-1},p'_{2n-1}) + d(Sp_{2n-1},Tp_{2n-2}) + q^{2n-1} .$$

Since $p_{2n-1} \in Q_1$ from (2) we obtain that:

$$d(Sp_{2n-1},Tp_{2n}) \leqslant d(Tp_{2n-2},p'_{2n-1}) + q^{2n-1} .$$

Further from $p_{2n-1} \in Q_1$ it follows that $p_{2n-2} \in P_o$ which means that $Tp_{2n-2} = p'_{2n-2}$.

Hence:

$$d(Sp_{2n-1},Tp_{2n}) \leqslant d(p'_{2n-2},p'_{2n-1}) + q^{2n-1} \leqslant$$

$$\leqslant qd(Sp_{2n-3},Tp_{2n-2}) + q^{2n-2} + q^{2n-1} .$$

Using the above inequalities we conclude that (3) and (4) hold. Inequalities (3) and (4) can be written in the form:

$$z_{2n} = Tp_{2n}, z_{2n+1} = Sp_{2n+1} \quad (n \in N)$$

$$d(z_n,z_{n+1}) \leqslant \begin{cases} qd(z_n,z_{n-1}) + q^n \\ \text{or} \\ qd(z_{n-2},z_{n-1}) + q^n + q^{n-1} , \end{cases}$$

which is inequality (*) from [1].

Then: $$d(z_n,z_{n+1}) \leqslant q^{\frac{n}{2}} (\delta + n) ,$$

where

$$\delta = q^{-\frac{1}{2}} \max\{d(z_o,z_1),d(z_1,z_2)\} .$$

Hence the sequence $\{z_n\}_{n \in N}$ is a Cauchy sequence and let

$$z = \lim_{n\to\infty} z_n = \lim_{n\to\infty} Tp_{2n} = \lim_{n\to\infty} Sp_{2n+1} .$$

There exists at least one subsequence $\{p_{2n_k}\}_{k \in N}$ or $\{p_{2n_k+1}\}_{k \in N}$ which is contained in P_o or Q_o respectively since

$$(p_{2n},p_{2n+1}) \notin P_1 \times Q_1 \quad \text{and} \quad (p_{2n-1},p_{2n}) \notin Q_1 \times P_1 .$$

Suppose that there exists $\{p_{2n_k}\}_{k \in N}$ such that $p_{2n_k} \in P_o$ for every $k \in N$. Then

$$p'_{2n_k} = Tp_{2n_k} \in fp_{2n_k-1} \quad (k \in N) .$$

Let us prove that $Sz \in fz$. Using the weak commutativity of the pair (f,S) and the relations:

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$$Tp_{2nk} \in fp_{2nk-1} \cap K \quad \text{and} \quad Sp_{2nk-1} \in K, \quad k \in N$$

we obtain that:

$$d(STp_{2nk}, fSp_{2nk-1}) \leqslant d(fp_{2nk-1}, Sp_{2nk-1}) \leqslant$$

$$\leqslant d(Tp_{2nk}, Sp_{2nk-1}) .$$

Since $\lim_{k\to\infty} d(Tp_{2nk}, Sp_{2nk-1}) = 0$, we obtain that

$$\lim_{k\to\infty} d(STp_{2nk}, fSp_{2nk-1}) = 0 .$$

From the inequality:

$$d(STp_{2nk}, fz) \leqslant d(STp_{2nk}, fSp_{2nk-1}) + H(fSp_{2nk-1}, fz) ,$$

since f is H-continuous, we obtain that:

$$\lim_{k\to\infty} d(STp_{2nk}, fz) = 0 .$$

Hence from the inequality:

$$d(Sz, fz) \leqslant d(Sz, STp_{2nk}) + d(STp_{2nk}, fz)$$

using the continuity of the mapping S we obtain that $d(Sz, fz) = 0$ and so $Sz \in fz$, which implies that $\{Tz, Sz\} \cap fz \neq \emptyset$.

Suppose now that $S, T: M \to M$ and that *(i)* and *(ii)* hold. From 4. we obtain that for every $k \in N$:

$$d(p'_{2nk+1}, p'_{2nk}) \leqslant H(fp_{2nk}, fp_{2nk-1}) + q^{2nk} \leqslant$$

$$\leqslant qd(Tp_{2nk}, Sp_{2nk-1}) + q^{2nk}$$

which implies that:

$$d(p'_{2nk+1}, Tp_{2nk}) \leqslant qd(Tp_{2nk}, Sp_{2nk-1}) + q^{2nk} .$$

Since $\lim_{k\to\infty} d(Tp_{2nk}, Sp_{2nk-1}) = 0$, we obtain that:

$$\lim_{k\to\infty} p'_{2nk+1} = z$$

and so :

$$Tz = T(\lim_{k\to\infty} p'_{2nk+1}) = \lim_{k\to\infty} Tp'_{2nk+1} . \tag{5}$$

Using the implication *(i)* for $x = p_{2nk}$ and $y = p'_{2nk+1}$ we conclude that:

$$d(Tp'_{2nk+1}, fTp_{2nk}) \leqslant d(fp_{2nk}, Tp_{2nk}) \leqslant d(p'_{2nk+1}, Tp_{2nk})$$

and since $\lim_{k\to\infty} d(p'_{2nk+1}, Tp_{2nk}) = d(z,z) = 0$, we have that

$$\lim_{k\to\infty} d(Tp'_{2nk+1}, fTp_{2nk}) = 0 .$$

Further:

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$$d(Tp'_{2nk+1},fz) \leq d(Tp'_{2nk+1},fTp_{2nk}) + H(fTp_{2nk},fz)$$

and since $\lim_{k\to\infty} H(fTp_{2nk},fz) = 0$, it follows that:

(6) $$\lim_{k\to\infty} d(Tp'_{2nk+1},fz) = 0 .$$

Using (5) and (6) we obtain that $d(Tz,fz) = 0$ since:

$$d(Tz,fz) \leq d(Tz,Tp'_{2nk+1}) + d(Tp'_{2nk+1},fz) .$$

From $d(Tz,fz) = 0$ we conclude that $Tz \in fz$.

COROLLARY (THEOREM A) [1] *Let* (M,d) *be a complete convex metric space,* K *a nonempty closed subset of* M, $f:K\to CB(M)$ *so that for every* $x \in \partial K$, $fx \subset K$ *and:*

$$H(fx,fy) \leq qd(x,y) , \text{ for every } x,y \in K$$

where $q \in (0,1)$. *Then there exists* $z \in K$ *such that* $z \in fz$.

PROOF: It is obvious that for $S = T = Id|K$ all the conditions of Theorem 1 are satisfied.

3. A COMMON FIXED POINT THEOREM IN METRIC SPACES WITH A CONVEX STRUCTURE

In this section we shall need the following definition which is given in [5].

DEFINITION 1. *Let* (X,d) *be a metric space,* $A,S:X\to 2^X$ *and* $K\subseteq X$. *The mapping* A *is said to be* (α,S)*-densifying on the set* K *if and only if for every* $M\subseteq K$ *such that* $S(M),A(M) \in B(X)$ *the following implication holds:*

$$\alpha(S(M)) \leq \alpha(A(M)) \Longrightarrow \overline{M} \text{ is compact.}$$

REMARK In [4] the definition of the Palais-Smale condition for the singlevalued mappings is given. We shall recall this definition. Let X and Γ be metric spaces and $p,q : \Gamma\to X$. The pair (p,q) satisfies the Palais-Smale condition if for every sequence $\{y_n\}_{n\in N}$ from Γ the relation

$$\lim_{n\to\infty} d_X(p(y_n),q(y_n)) = 0$$

implies that there exists a convergent subsequence $\{y_{n_k}\}_{k\in N}$ of the sequence $\{y_n\}_{n\in N}$.

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From this definition it is obvious that if in Definition 1. the mappings S and A are singlevalued and A(K) is bounded, the following implication holds:

A is (α,S) densifying on $K \Rightarrow$ the pair (A,S) satisfies the Palais-Smale condition on K (i.e. the pair $(A|_K, S|_K)$ satisfies the Palais-Smale condition, where $A|_K$ and $S|_K$ are the restrictions of A and S on K respectively).

Indeed, if A is (α,S) densifying on K and A(K) is bounded then from:

$$\lim_{n\to\infty} d_X(S(y_n), A(y_n)) = 0$$

it follows that $\alpha(SL) = \alpha(AL)$ (this can be proved easily), where $L = \{y_n, n \in N\}$, which implies that $\overline{L}$ is compact. Hence the pair $(A|_K, S|_K)$ satisfies the Palais-Smale condition.

Let us remark that in [4] the notion of a condensing pair (p,q) of singlevalued mappings is given, which is similar to Definition 1 if the mappings A and S are singlevalued. If $q : \Gamma \to 2^X$ and $p : \Gamma \to X$, we can introduce the Palais-Smale condition in the following way.

DEFINITION 2. *The pair* (p,q) *satisfies the Palais-Smale condition if for every sequence* $\{y_n\}_{n\in N}$ *from* Γ, *the relation:*

$$\lim_{n\to\infty} d_X(p(y_n), v_n) = 0, \text{ for some } \{v_n\}_{n\in N}$$

such that $v_n \in q(y_n)$ $(n \in N)$, *implies the existence of a convergent subsequence* $\{y_{n_k}\}_{k\in N}$ *of the sequence* $\{y_n\}_{n\in N}$.

The set of coincidence points for the pair (p,q) is defined by:

$$K(p,q) = \{y,\ y \in \Gamma \text{ and } p(y) \in q(y)\}.$$

and for $\varepsilon > 0$ we shall define an ε-coincidence point for the pair (p,q) in the following way.

DEFINITION 3. *A point* $y \in \Gamma$ *is an* ε*-coincidence point for the pair* (p,q) *if there exists an element* $v \in q(y)$ *such that* $d_X(p(y), v) < \varepsilon$.

Similarly as in [4] (Proposition 2.3)we can prove the following proposition.

PROPOSITION: *Let* (Γ, d_Γ) *and* (X, d_X) *be metric spaces*, $q : \Gamma \to 2^X$ *a closed mapping and* $p : \Gamma \to X$ *a continuous map-*

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ping. Let for every $\varepsilon > 0$ *the pair* (p,q) *have an* ε*-coincidence point and satisfy the Palais-Smale condition. Then*

$$K(p,q) \neq \emptyset .$$

PROOF: Let $\varepsilon_n = \frac{1}{n}$ $(n \in N)$ and y_n be an ε_n-coincidence point for the pair (p,q). Let $v_n \in q(y_n)$, $(n \in N)$ so that $d_X(p(y_n), v_n) < \frac{1}{n}$. Then

$$\lim_{n\to\infty} d_X(p(y_n), v_n) = 0$$

and since the pair (p,q) satisfies the Palais-Smale condition, there exists a convergent subsequence $\{y_{n_k}\}_{k\in N}$ of the sequence $\{y_n\}_{n\in N}$. Let $\lim_{k\to\infty} y_{n_k} = y$. Since $\lim_{k\to\infty} d_X(p(y_{n_k}), v_{n_k}) = 0$, it follows that

$$\lim_{k\to\infty} v_{n_k} = \lim_{k\to\infty} p(y_{n_k}) = p(y) .$$

From $v_{n_k} \in q(y_{n_k})$, $(k \in N)$, using the closedness of the mapping q, we obtain that $p(y) \in q(y)$, which means that $y \in K(p,q)$.

Similarly as in [5] we shall prove a common fixed point theorem if:

$$H(fx,fy) \leqslant d(Sx,Ty) \text{ , for every } x,y \in K ,$$

where K is (W,x_o)-star convex ($x_o \in K$ and $W(K,x_o,(0,1)) \subset K$)

THEOREM 2. *Let* (M,d) *be a complete metric space with a convex structure* W *with property* II, K *a nonempty closed* (W,x_o) *- star convex subset of* M, $S,T : M \to M$ *continuous* (W,x_o) *convex mappings,* $f : K \to k(M)$ *(the family of all nonempty compact subsets of* M*) so that the following conditions are satisfied:*

1. fK *is bounded,* $K \subseteq SK \cap TK$ *and:*
$$Tx \in \partial K \Longrightarrow fx \in K \; ; \quad Sx \in \partial K \Longrightarrow fx \in K .$$
2. *For every* $x,y \in K$:
$$H(fx,fy) \leqslant d(Sx,Ty) .$$
3. *The mapping* f *is* H*-continuous and:*
$$Sx \in K \Longrightarrow fSx = Sfx \; ; \quad Tx \in K \Longrightarrow Tfx = fTx .$$
4. *The mapping* f *is* (α,S) *or* (α,T) *densifying.*

Then there exists $z \in K$ *so that* $Tz \in fz$ *and* $Sz \in fz$.

PROOF: As in [5] let $\{k_n\}_{n\in N}$ be such a sequence from $(0,1)$ that $\lim_{n\to\infty} k_n = 1$ and $f_n : K \to k(M)$ be defined by:

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$f_n x = W(fx, x_o, k_n)$, for every $x \in K$ and every $n \in N$. We shall prove that for every $n \in N$ there exists $x_n \in K$ so that $Tx_n \in f_n x_n$ and $Sx_n \in f_n x_n$. It remains to be proved that T,S and f_n satisfy all the conditions of Theorem 1. Since the set K is (W,x_o)-star convex from the condition 1. of this theorem we obtain that

$$Tx \in \partial K \Rightarrow f_n x \in K; \quad Sx \in \partial K \Rightarrow f_n x \in K.$$

Further $f_n Sx$, if $Sx \in K$, is equal to the set $Sf_n x$ since:

$$f_n Sx = \bigcup_{z \in fSx} W(z, x_o, k_n) = \bigcup_{z \in Sfx} W(z, x_o, k_n) =$$

$$= \{W(Sy, x_o, k_n), y \in fx\} = \{SW(y, x_o, k_n), y \in fx\} = Sf_n x, \text{ for every } n \in N.$$

Similarly from $Tx \in K$ we obtain that $f_n Tx = Tf_n x$, for every $n \in N$. It is easy to see that from $f_n Sz = Sf_n z$ we obtain the inequality: $d(Sy, f_n Sz) \leqslant d(f_n z, Sz)$, for $y \in f_n z$ since

$$d(Sy, f_n Sz) = \inf_{z \in f_n Sz} d(Sy, z) = \inf_{z \in Sf_n z} d(Sy, z) <$$

$$\leqslant d(Sy, Sy) = 0 \leqslant d(f_n z, Sz).$$

Since the convex structure W satisfies condition II it follows that:

$$H(f_n x, f_n y) = H(W(fx, x_o, k_n), W(fy, x_o, k_n)) \leqslant$$

$$\leqslant k_n H(fx, fy) \leqslant k_n \, d(Sx, Ty)$$

for every $x, y \in K$, and hence the mapping f_n is H-continuous. Further, from the compactness of fx, for every $x \in K$ and the continuity of W in respect to the first variable it follows that $f_n x$ is compact for every $x \in K$. From $K \subseteq SK \cap TK$ we obtain that $f_n K \cap K \subseteq SK \cap TK$. Hence, all the conditions of Theorem 1. are satisfied and for every $n \in N$ there exists $x_n \in K$ so that $Tx_n \in f_n x_n$ and $Sx_n \in f_n x_n$. The rest of the proof is as in the proof of Theorem 2. from [5]. For the completeness we shall give the rest of the proof. Since $Tx_n \in f_n x_n$ and $Sx_n \in f_n x_n$ $(n \in N)$ there exist $u_m \in fx_m$ and $v_m \in fx_m$ $(m \in N)$ so that $Sx_m = W(u_m, x_o, k_m)$ and $Tx_m = W(v_m, x_o, k_m)$. Then it is easy to prove that:

$$\lim_{m \to \infty} d(Sx_m, u_m) = \lim_{m \to \infty} d(Tx_m, v_m) = 0. \tag{7}$$

Hence, for every $\varepsilon > 0$ there exists an ε-coincidence point for (S,f) and (T,f). Since the mapping f is (α,S) or (α,T) densi-

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fying on the set K and the set $f(K)$ is bounded, we conclude that the pair (S,f) or the pair (T,f) satisfies the Palais-Smale condition on the set K. Further, the mapping f is H-continuous and fx is compact for every $x \in K$ and so the mapping f is closed. Using the Proposition we conclude that $K(S,f) \neq \emptyset$ or $K(T,f) \neq \emptyset$. From the Proposition it follows that $y \in K(S,f)$ (if, for example $K(S,f) \neq \emptyset$) is of the form $y = \lim_{k\to\infty} x_{n_k}$, for some subsequence $\{x_{n_k}\}_{k\in N}$ of the sequence $\{x_n\}_{n\in N}$. From (7) we have that $\lim_{k\to\infty} v_{n_k} = Ty \in fy$ and so

$$y \in K(T,f) \cap K(S,f) .$$

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REZIME

TAČKE KOINCIDENCIJE VIŠEZNAČNIH PRESLIKAVANJA
U KONVEKSNIM METRIČKIM PROSTORIMA

U ovom je radu dokazano uopštenje dobro poznate teoreme o nepokretnoj tački Assada i Kirka za višeznačna preslikavanja u konveksnim metričkim prostorima. Višeznačna verzija uslova Palais-Smalea je uvedena i primenjena u dokazu Teoreme 2, koja sadrži jedan rezultat o postojanju tačke koincidencije za višeznačna preslikavanja u metričkim prostorima sa konveksnom strukturom.

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SOME FIXED POINT THEOREMS FOR MULTIVALUED MAPPINGS IN BANACH SPACES AND THEIR APPLICATIONS

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ABSTRACT

In this paper a multivalued generalization of the well known Melvin fixed point theorem [16] is obtained. Using this generalization an existence result for a class of functional equations is obtained.

INTRODUCTION

For functional-differential equations a tool that has been widely used to prove existence of solutions has been the fixed point method . The aim of this paper is to obtain multivalued generalizations of the well known Melvin fixed point theorem [16] and to apply these results on the existence problem for functional equations of the form:

$$(1)\quad g(t)\in f(t,(Tg)(t),\int_0^t g(u)du) \qquad t\in I=[0,a].$$

An existence result for the equation $x(t)\in F(t,x)$ is also obtained and may be used in the proof of the existence of a solution of functional equations of the form:

$$x(t)\in f(t,\int_0^{s(t)} g(t,u,x(u))du, x(r(t)), \quad t\in I$$

where $r:I\to I$ and $s:I\to I$ are continuous mappings .

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Some multivalued generalizations of Melvin's fixed point theorem are proved in [10], [11], [12], [20] and [21].

In [16] Melvin proved the following fixed point theorem:

Let (X,d) be a metric space, A a nonempty, closed and convex subset of a Banach space $(Y,\|\ \|)$, $T:A\to X$ a continuous mapping such that $\overline{T(A)}$ is compact in X and $G:\overline{T(A)}\times A\to A$ so that the following conditions are satisfied:

(i) $G(\cdot,u):\overline{T(A)}\to A$ is continuous, for each fixed $u\in A$.

(ii) $\|G(x,u)-G(x,v)\| \leq q\|u-v\|$, for each $x\in\overline{T(A)}$

and each $u,v\in A$, where $q\in[0,1)$.

Then there exists $x\in A$ such that $x = G(Tx,x)$.

The idea of the proof of the above fixed point theorem is the following. First, the author proved the existence of the continuous mapping $f:\overline{T(A)}\to A$ such that for every $x\in\overline{T(A)}$ $f(x) = G(x,f(x))$ and then, applying the Schauder fixed point theorem to the mapping Q defined by $Qx = f(Tx)$ $(x\in A)$, the assertion in the above theorem follows.

Hence, the main problem in the multivalued case of the mapping G is to prove the existence of a continuous mapping $f:\overline{T(A)}\to A$ such that for every $x\in\overline{T(A)}$: $f(x)\in G(x,f(x))$. If we denote by F the multivalued mapping defined by :

$$F(u) = \{x \mid x\in A, x\in G(u,x)\}\ (u\in\overline{T(A)})$$

it is obvious that the mapping f (if it exists) is, in fact, a continuous selector of the mapping F. Let us remark that the upper semicontinuity of the mapping F is investigated in Cheng's paper [3].

Using a result of Banks and Jacobs, which holds only for uniformly convex Banach spaces, in [12] the following theorem is obtained :

THEOREM A Let X be a metric space, A a nonempty, closed and convex subset of a uniformly convex Banach space $(Y,\|\ \|)$, $T:A\to X$ a continuous mapping such that $T(A)$ is compact in X and $G:T(A)\times A\to CCB(A)$ (the family of all nonempty, closed convex and bounded subsets of A) such that the following conditions are satisfied:

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1. $G(.,u):\overline{T(A)} \to CCB(A)$ is continuous for each fixed $u \in A$.

2. If D is the Hausdorff metric in Y then :

$$D(G(x,u),G(x,v)) \leq h(\|u-v\|)$$

for each $x \in T(A)$ and $u,v \in A$ where $h:[0,\infty) \to [0,\infty)$ is a nondecreasing mapping such that $\sum_{n \in N} h^n(t) < \infty \ (t>0)$.

Then there exists $x \in A$ such that $x \in G(Tx,x)$.

If $h(t)=kt \ (t \geq 0)$, where $k \in (0,1)$ and Y is a Hilbert space Theorem A is proved by Kisielewicz in [11] and if Y is a Banach space by Rzepecki in [21] . A generalization of Theorem A is given in [20] .

Using an another approach (Michael's selection theorem) we prove a multivalued generalization of Melvin's fixed point theorem ,where Y is an arbitrary Banach space and the mapping h satisfies some different conditions than in Theorem A.

Theorem 1 of this paper can be generalized to the case of a complete paranormed space Y (such a space is $S(I,R^n)$ - the space of all classes of Lebesgue measurable functions from $I=[0,a]$ into R^n) by using Zima's fixed point theorem and a generalization of Michael's selection theorem recently obtained by the authors and Lj.Gajić in [10].

1. PRELIMINARIES

Let $h:[0,\infty) \to [0,\infty)$ and introduce the following conditions as in [28] :

(A_1) h is right continuous .

(A_2) For each $q > 0$:

$$\sup\{u \mid u \leq h(u)+q\} = m(q) < \infty .$$

(A_3) For each $q > 0$: $\lim_{n \to \infty} h^n(m(q)) = 0$.

(A_4) For any $u \geq 0$ and $v \geq 0$: $h(u)+h(v) \leq h(u+v)$.

From [28] we have the following fixed point theorem.

THEOREM B Suppose that for every $x,y \in X$, where (X,d) is a complete metric space, the following condition holds:

$$D(Tx,Ty) \leq h(d(x,y))$$

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where $T:X \to CB(X)$ (the family of all nonempty, closed and bounded subsets of X) is continuous and h satisfies (A_1)-(A_4). Then there exists $x \in X$ such that $x \in Tx$.

Remark: It is obvious that (A_4) implies that h is a nondecreasing mapping . Further, if h is a nondecreasing mapping such that $\lim_{n\to\infty} h^n(q) = 0$,for each $q > 0$ and $\lim_{q\to\infty} (q-h(q)) = \infty$ the condition (A_2) holds [18] .Condition (A_4) is satisfied, for example ,for $h(t) = q(t)t$,where q is a nondecreasing mapping .

We shall also use in the next text the following fixed point theorem [19] , [24] .

THEOREM C Let (X,d) be a complete metric space and $T:X \to CB(X)$ so that for every $x,y \in X$:

$$D(Tx,Ty) \leq q(d(x,y))d(x,y)$$

where $q: [0, \infty) \to [0,1)$ is upper semicontinuous from the right . Then there exists $x \in X$ such that $x \in Tx$.

Let E be a linear space over the real or complex number field . The function $q:E \to [0,\infty)$ is said to be a paranorm if and only if :

1. $q(x) = 0 \iff x = 0$.
2. $q(-x) = q(x)$,for every $x \in E$.
3. $q(x+y) \leq q(x)+q(y)$,for every $x,y \in E$.
4. If $q(x_n-x_o) \to 0$ and $r_n \to r_o$ (r_n, r_o from the scalar field) then $q(r_n x_n - r_o x_o) \to 0$.

Then (E,q) is a paranormed space, which is also a topological vector space in which the family of neighbourhoods of zero is given by $V = \{V_r\}_{r>0}$,where $V_r = \{ x \mid x \in E , q(x) < r \}$. In [27] K.Zima proved a generalization of Schauder's fixed point theorem for mapping $T:K \to K (K \subset E)$,where K satisfies the following condition (in the next text the Zima condition) :

$$q(rx) \leq C(K)\, r\, q(x) \quad \text{,for every } r \in [0,1]$$

and every $x \in K-K$. Here, C(K) is a constant which depends on K .

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Let us give an example of a subset which satisfies the Zima condition .

Let $a > 0$, $I = [0,a]$ and for $x \in S(I,R^n)$ (the space of all equivalence classes of Lebesgue measurable functions, defined on I with values in R^n) , $q(x)$ is defined by :

$$q(x) = \int_I \frac{|x(t)|_{R^n}}{1 + |x(t)|_{R^n}} dt \quad , \{x(t)\} \in x .$$

It is easy to see that q is a paranorm . Let $s > 0$ and $y \in S(I,R^n)$. Further ,let $K = y + K_s$,where K_s is defined by :

$$K_s = \{x \mid x \in S(I,R^n) , |x(t)|_{R^n} \leq s , t \in I\} .$$

Then [8] we have that :

$$q(r(x_1-x_2)) \leq (1+2s)r\, q(x_1-x_2)$$

for every $x_1, x_2 \in K_s$ and every $r \geq 0$. This means that $C(K_s) = 1 + 2s$ and so $C(K) = 1 + 2s$. Hence $C(K) = C(K_s) = 1+2s$.

REMARK: Let E be a topological vector space, U the family of all neighbourhoods of zero in E and K a nonempty subset of E . Motivated by Zima's condition ,in [8] we introduced the following definition: A subset K of E is of Zima's type if and only of for every $V \in U$ there exists $U \in U$ so that $co(U \cap (K-K)) \subset V$ (co is the operator of the convex hull) .

It is easy to see that if K satisfies the Zima condition it is of Zima's type . It is proved in [8] that every subset of Zima's belongs to the class of admissible subsets of a topological vector space [9] . The class of subsets of Zima's type has many important properties which are useful in the fixed point theory [9] .

THEOREM D Let X be a paracompact topological space,(Y,q) a complete paranormed space,$F: X \to CC(Y)$(the family of all convex closed nonempty subsets of Y) a lower semicontinuous mapping such that $K = co\, F(X)$ (the convex hull of $F(X)$) satisfies the Zima condition . Then there exists a continuous selection of the mapping F.

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2.EXISTENCE RESULTS FOR THE EQUATION $x \in G(Tx,x)$

First,we shall prove the following lemma .

LEMMA 1 Let (X,d) be a metric space , A a nonempty closed and convex subset of a Banach space $(Y,\|\ \|)$, $T:A\to X$ a continuous mapping such that $\overline{T(A)}$ is compact in X and $G:\overline{T(A)}\times A \to CCB(A)$ is such that the following conditions are satisfied:

(i) $G(.,u):\overline{T(A)}\to CCB(A)$ is continuous ,for each fixed $u\in A$.

(ii) $D(G(x,u),G(x,v)) \leqslant h(\|u-v\|)$,for each $x\in\overline{T(A)}$ and $u\in A, v\in A$,where $h:[0,\infty)\to[0,\infty)$ satisfies $(A_1),(A_2),(A_4)$ and $\lim\limits_{n\to\infty} h^n(q) = 0$,for every $q>0$.

Then there exists a continuous mapping $f:\overline{T(A)}\to A$ so that

$$f(x)\in G(x,f(x)) \quad \text{,for every } x\in\overline{T(A)} .$$

Proof: Let $U=\{g \mid g:\overline{T(A)}\to Y$,g is continuous $\}$ and for every $g_1,g_2\in U$ let :

$$r(g_1,g_2) = \sup_{x\in\overline{T(A)}} \|g_1(x)-g_2(x)\| .$$

Then (U,r) is a complete metric space and let the mapping F be defined in the following way:

$$F(g) = \{u \mid u\in U ,u(x)\in G(x,g(x)) ,x\in\overline{T(A)}\} \quad \text{,for } g\in U .$$

As in [12] (Lemma 2 , which holds for any Banach space Y) it follows that the mapping $x\mapsto G(x,g(x))$ $(x\in\overline{T(A)})$ is continuous for every $g\in U$ and so from Michael's selection theorem [17] it follows that $F(g)\neq\emptyset$,for every $g\in U$. Let us prove that for every $g_1,g_2\in U$ we have :

(2) $$D_r(F(g_1),F(g_2)) \leqslant h(r(g_1,g_2))$$

where D_r is the Hausdorff metric induced by r .

In order to prove (2) we shall prove that for every $v\in F(g_1)$:

(3) $$d_r(v,F(g_2)) \leqslant h(r(g_1,g_2))$$

and for every $s\in F(g_2)$:

(4) $$d_r(F(g_1),s) \leqslant h(r(g_1,g_2)) .$$

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Here $d_r(v,F(g_2))= \inf_{z\in F(g_2)} r(v,z)$. Let $t>0$. We shall prove the existence of $w\in F(g_2)$ so that $r(v,w)< h(r(g_1,g_2))+t$ which implies (3).

From the definition of the Hausdorff metric D it follows that for every $x\in\overline{T(A)}$ there exists $y_x\in A$ such that $y_x\in G(x,g_2(x))$ and:

(5) $$\|v(x)-y_x\| < D(G(x,g_1(x)),G(x,g_2(x)))+ t/2 .$$

From (ii) and (5) it follows that:

$$\|v(x)-y_x\| < h(\|g_1(x)-g_2(x)\|)+t/2$$

and since h is nondecreasing and $\|g_1(x)-g_2(x)\| \leq r(g_1,g_2)$ we obtain that :

$$\|v(x)-y_x\| < h(r(g_1,g_2)) + t/2$$

From Michael selection theorem and Example 1.3 [17] it follows that for every $x\in\overline{T(A)}$ there exists a selector of the mapping $z\to G(z,g_2(z))(z\in\overline{T(A)})$,denoted by $w_x:\overline{T(A)}\to A$ such that $w_x(x)=y_x$. Since w_x is a selector of the mapping $z\to G(z,g_2(z))$ we have that $w_x(z)\in G(z,g_2(z))$,for every $z\in\overline{T(A)}$. Since v and w are continuous ,there exists an open neighbourhood U_x of x so that

(6) $$\|v(z)-w_x(z)\| < h(r(g_1,g_2)) + t/2$$

for every $z\in U_x$.The set $\overline{T(A)}$ is paracompact(as a metric space) and s there exists a locally finite partition of unity P subordinated to covering $\{U_x\}_{x\in\overline{T(A)}}$. Let $P=\{p_s\}_{s\in A}$ and suppose that supp $p_s\subset U_{x_s}$. This means that $p_s(x)\neq 0$ implies that $x\in U_{x_s}$.

Let $w:\overline{T(A)}\to A$ be defined in the following way:

$$w(z) = \sum_{s\in A} p_s(z)w_{x_s}(z) \quad ,z\in\overline{T(A)} .$$

Since p_s and w_{x_s} are continuous and P is a locally finite partition of unity it follows that w is continuous and from $w_{x_s}(z)\in G(z,g_2(z)),z\in\overline{T(A)}$ and the convexity of $G(z,g_2(z))$ it follows that $w(z)\in G(z,g_2(z)),z\in\overline{T(A)}$.

Let us prove that for every $z\in\overline{T(A)}$:

$$\|v(z)-w(z)\| < h(r(g_1,g_2)) + t/2 .$$

We have that :

$$\|v(z)-w(z)\| = \|\sum_{s\in A} p_s(z)v(z)- \sum_{s\in A} p_s(z)w_{x_s}(z)\| =$$

$$= \|\sum_{s\in A} p_s(z)\,[v(z)-w_{x_s}(z)]\| =$$

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$$= \| \sum_{s:p_s(z)\neq 0} p_s(z)\,[v(z)-w_{x_s}(z)] \|$$

$$< \sum_{s:p_s(z)\neq 0} p_s(z)\,\|v(z)-w_{x_s}(z)\| \quad .$$

Since from $p_s(z)\neq 0$ we obtain that $z\in U_{x_s}$ it follows that $p_s(z)\neq 0$ implies that $\|v(z)-w_{x_s}(z)\| < h(r(g_1,g_2))+t/2$ and so (6) implies that:

$$\|v(z)-w(z)\| < \sum_{s:p_s(z)\neq 0} p_s(z)\,[h(r(g_1,g_2)) + t/2\,]$$

$$= h(r(g_1,g_2)) + t/2 \quad .$$

Hence we have that :

$$r(v,w) \leq h(r(g_1,g_2)) + t/2 < h(r(g_1,g_2)) + t$$

and so (3) is proved . Similarly we can prove (4) .

From Theorem B it follows that there exists $f\in U$ so that $F(f)$ contains f which means that $f(x)\in G(x,f(x))$,for every $x\in\overline{T(A)}$.

Remark: A similar method as in the proof of Lemma 1 is used in the proof of Theorem 25.2 in [4] . If $h(t) = q(t)t$, where q is a nondecreasing and upper semicontinuous from the right function from $[0,\infty)$ into $[0,1)$ we can apply Theorem C in order to prove the existence of $f\in U$ such that $f\in F(f)$. If $h:[0,\infty)\to[0,\infty)$ is upper semicontinuous and strictly increasing function such that $\sum_{n\in N} h^n(t)<\infty$ $(t>0)$ from Corollary 2 [25] it follows the existence of $f\in U$ such that $f\in F(f)$. Further,if Y is a uniformly convex Banach space and $h(t)=q(t)t$ $(t\geq 0)$,where $q:[0,\infty)\to[0,1)$ it is enough to suppose that q is only a nondecreasing function . Namely , in this case $h^n(t)\leq (q(t))^n t$ and so $\sum_{n\in N} h^n(t)<\infty$ and we can apply the Lemma from [12] .

THEOREM 1 Let all the conditions of Lemma 1 be satisfied. Then there exists $x\in A$ so that $x\in G(T(x),x)$.

Proof:Let $Qx = f(Tx)$,$x\in A$,where $f\in U$ is from Lemma 1 . From the continuity of f and the compactness of $\overline{T(A)}$ it follows that the mapping $Q:A\to A$ satisfies all the conditions of th

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Schauder fixed point theorem and so there exists $x_o \in A$ such that $x_o \in Qx_o$. Hence, $x_o \in G(T(x_o),x_o)$.

LEMMA 2. Let $(Y,\|\ \|)$ be a Banach space, A a nonempty, closed and convex subset of Y, M a metric space, $G:M\times A \to CCB(A)$, $h:[0,\infty)\to[0,\infty)$ a nondecreasing mapping such that $h(t)<t$ $(t>0)$ and so that the following conditions are satisfied:

1. For every $y\in A$, $x \mapsto G(x,y)(x\in M)$ is a continuous mapping.
2. There exist $y_o \in A$, $r>1$ and $t \geq \sup\{ d(y_o,G(x,y_o)) | x\in M \}$ so that the sequence $\{t_n\}_{n\in N\cup\{0\}}$ defined by :

$$t_{n+1}=t_n+h(r(t_n-t_{n-1})) \ , n\in N \ , t_o=0, t_1=t$$

is bounded .

3. For every $x\in M$ and $y_i \in A (i\in\{1,2\})$:

$$D(G(x,y_1),G(x,y_2)) \leq h(\|y_1-y_2\|) \ .$$

If M is compact or A is bounded then there exists a continuous mapping $f:M\to A$ such that $f(x)\in G(x,f(x))$, $x\in M$.

Proof: Since $r>1$ it follows from Lemma 7.1 [17] that there exists $f_1:M\to A$ which is continuous and such that :

$$\|f_1(x)-y_o\| = r\, d(y_o,G(x,y_o)) \text{ and } f_1(x)\in G(x,y_o)$$

for every $x\in M$. Let $y_o = f_o(x)$, for every $x\in M$.

It is easy to see that for every $n\in N$ there exists a continuous mapping $f_n:M\to A$, such that $f_n(x)\in G(x,f_{n-1}(x))$, for every $x\in M$ and :

$$\|f_n(x)-f_{n-1}(x)\| = r\, d(f_{n-1}(x),G(x,f_{n-1}(x))) \ .$$

Let us prove that for every $n\in N$:

$$\|f_n(x)-f_{n-1}(x)\| \leq r(t_n-t_{n-1}) \text{ for every } x\in M \ .$$

For $n=1$ we have :

$$\|f_1(x)-f_o(x)\| = \|f_1(x)-y_o\| = r\, d(y_o,G(x,y_o))$$

$$\leq r \sup_{x\in M} d(y_o,G(x,y_o)) \leq r(t_1-t_o) \ .$$

Suppose that $\|f_n(x)-f_{n-1}(x)\| \leq r(t_n-t_{n-1})$. Then

$$\|f_{n+1}(x)-f_n(x)\| \leq rD(G(x,f_n(x)),G(x,f_{n-1}(x))) \leq$$

$$\leq r\, h(\|f_n(x)-f_{n-1}(x)\|) \leq r\, h(r(t_n-t_{n-1})) = r(t_{n+1}-t_n) \ .$$

From this we obtain that:

$$\|f_{n+m}(x)-f_n(x)\| \leq \sum_{k=n+1}^{n+m} \|f_k(x)-f_{k-1}(x)\| \leq r \sum_{k=n+1}^{n+m} (t_k-t_{k-1})$$

$$= r(t_{n+m}-t_n) \ .$$

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Since $\{t_n\}_{n\in N\cup\{0\}}$ is a monotone nondecreasing sequence such that $t_0=0<t_1\leq t_2\leq \ldots \leq a$,it follows that there exists t^* such that $\lim_{n\to\infty} t_n = t^*$. Hence f_n is a uniformly convergent sequence of functions and let $f(x) = \lim_{n\to\infty} f_n(x)$. As in [12] it follows that $f(x)\in G(x,f(x))$, $x\in M$.

THEOREM 2 Let (X,d) be a metric space, A a nonempty,closed and convex subset of a Banach space $(Y,\|\ \|)$, $T:A\to X$ a continuous mapping such that $\overline{T(A)}$ is compact in X $h:[0,\infty)\to[0,\infty)$ a nondecreasing mapping such that $h(t)<t$ $(t>0)$,and $G:\overline{T(A)}\times A\to CCB(A)$ is such that 1.,2. and 3. in Lemma hold for $M=\overline{T(A)}$.

Then there exists $x_0\in A$ so that $x_0\in G(Tx_0,x_0)$.

Proof:From Lemma 2,as in Theorem 1 ,we conclude that mapping $Q:A\to A$,$Qx = f(Tx)$ ($x\in A$) has a fixed point $x_0\in Qx_0$.

Remark: If $h(u) = k\,u (u>0)$ and $k\in(0,1)$ then for every $r>1$ such that $kr<1$ it follows that :

$$t_{n+1} = (1+kr+k^2r^2+\ldots+k^nr^n)\,t \quad ,n\in N$$

is bounded for every $t>0$. Hence ,Theorem 2 is a generalization of Rzepecki's fixed point theorem from [21] .

Theorems 1. and 2. can be generalized to paranormed spaces .

From Theorem D ,using Example 1.3 from [17] we obtain the following result .

PROPOSITION Let X be a paracompact topological space,(Y,q) a complete paranormed space,$F:X\to CC(Y)$ a lower semicontinuous mapping and A a closed subset of X . If co F(X) satisfies the Zima condition then every continuous selection $g:A\to Y$ of $F|A$ has an extension $f:X\to Y$ which is a continuous selection of F .

Proof: Let us define the mapping $H:X\to CC(Y)$ by :

$H(x) = g(x)$,for every $x\in A$ and $H(x) = F(x)$,for every $x\in X\setminus A$. Then [17] H is lower semicontinuous and co $H(X)\subset$ co $F(X)$ which implies that co H(X) satisfies the Zima condition . Then from Theorem D it follows that there exists a continuous selection $f:X\to Y$ of the mapping H ,

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which is obviously a continuous selection of F .

Now,we shall prove a generalization of Theorem 1 for paranormed spaces.In the next Theorem $C = C(\text{co } G(\overline{T(A)},A))$.

THEOREM 3 Let (X,d) be a metric space,A a nonempty,closed and convex subset of a complete paranormed space (Y,q) and satisfy the Zima condition,$T:A\to X$ a continuous mapping such that $\overline{T(A)}$ is compact in X ,$G:\overline{T(A)}\times A\to A$ and the following conditions are satisfied :

(i)' The condition (i) in Lemma 1 .

(ii)' For every $x\in\overline{T(A)}$ and every $u,v\in A$:

$$D(G(x,u),G(x,v))\leqslant h(q(u-v))$$

where $h:[0,\infty)\to[0,\infty)$ is such that the mapping $\bar{h}$ defined by $\bar{h}(t) = C\,h(t)$,for every $t\geqslant 0$,satisfies $(A_1),(A_2),(A_4)$ and $\lim_{n\to\infty}\bar{h}^n(a) = 0$,for every $a>0$.

Then there exists $x_o\in A$ such that $x_o\in G(Tx_o,x_o)$.

Proof: Since the set A satisfies the Zima condition and A is a convex set it follows that the set $\text{co } G(\overline{T(A)},A)$,which is contained in the set A ,satisfies the Zima condition also. Similarly as in Lemma 1 ,we can prove the existence of the continuous mapping $f:\overline{T(A)}\to A$ such that $f(x)\in G(x,f(x))$,for every $x\in\overline{T(A)}$. Namely,since we can apply the Proposition we obtain on the right side of (6) $C(h(r(g_1,g_2)) + \frac{t}{2})$,which implies that $D_r(F(g_1),F(g_2))\leqslant C\,h(r(g_1,g_2))$,for every $g_1,g_2\in U$.

From Theorem B it follows the existence of the mapping f and from the Zima generalization of the Schauder fixed point theorem [27] we obtain that there exists $x_o\in A$ such that $x_o\in G(Tx_o,x_o)$.

Let us give an application of Theorem 1 on the existence of a solution of functional equation (1) .

First,we shall give some notations .

$I = [0,a]$ $(a>0)$, $f:I\times R^n\times R^n\to CCB(R^n)$, $m\in L^1(I,R^n) =$

$$=\{x\mid x\in S(I,R^n)\ ,\ \int_I |x(t)|_{R^n}\,dt<\infty\ \}\quad ,A =\{w\mid w\in L^1(I,R^n),$$

$|w(t)|\leqslant m(t)$ a.e. on I} ,$T:A\to S(I,R^n)$ a compact mapping and $k:[0,a]\times R_+\to R_+$ so that $k(\cdot,s)\in L^1(I,R_+)$ $(s\in R_+)$.

THEOREM 4 Suppose that the mappings f and k satisfy the following conditions :

1. For every $(u,v)\in R^n\times R^n$,$t\mapsto f(t,u,v)$ is a measurable

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mapping.

2. For every $(t,u,v)\in I\times R^n\times R^n$, $D(f(t,u,v),\{\theta\})\leq m(t)$.

3. For every $(t,u,v_i)\in I\times R^n\times R^n$ $(i\in\{1,2\})$:

$$D(f(t,u,v_1),f(t,u,v_2))\leq k(t,|v_1-v_2|_{R^n}) .$$

4. For every $t\in[0,a]$, $k(t,\cdot)$ is a nondecreasing mapping such that the mapping $h:R_+\to R_+$, defined by $h(s)=\int_0^a k(t,s)dt$ satisfies all the conditions of Lemma 1 .

Then there exists a solution $g\in L^1(I,R^n)$ of the functional equation (1) .

Proof: Let $Y=L^1(I,R^n)$ and for every $x\in S(I,R^n)$ and $y\in A$

$G(x,y)=\{u|u:I\to R^n$, u is measurable , $u(t)\in f(t,x(t),\int_0^t y(u)du)$, $t\in I\}$

Then [14] $G(x,y)\neq\emptyset$, for every $x\in S(I,R^n)$ and every $y\in A$ and $G(x,y)\in CCB(A)$. Further, the set A is closed and convex and $G:\overline{T(A)}\times A\to A$.

Let us prove that for every $x\in S(I,R^n)$ and $y_i\in A(i\in\{1,2\})$

$$(7)\quad D_{L^1(I,R^n)}(G(x,y_1),G(x,y_2))\leq h(|y_1-y_2|_{L^1(I,R^n)}) .$$

Let $w_1\in G(x,y_1)$. Then (see , for example [4] , Lemma 23.2 there exists a measurable selector w_2 of the mapping $t\mapsto f(t,x(t),\int_0^t y_2(u)du)$ so that for every $t\in I$:

$$|w_1(t)-w_2(t)|_{R^n}=D(f(t,x(t),\int_0^t y_1(u)du),f(t,x(t),\int_0^t y_2(u)du)) .$$

From this we have that :

$$|w_1(t)-w_2(t)|_{R^n}\leq k(t,|\int_0^t[y_1(u)-y_2(u)]du|_{R^n})$$

$$\leq k(t,|y_1-y_2|_{L^1(I,R^n)})$$

and so $|w_1-w_2|_{L^1(I,R^n)}\leq\int_I k(t,|y_1-y_2|_{L^1(I,R^n)})dt=h(|y_1-y_2|_{L^1}$

Similarly, it can be proved that for every $w_2'\in G(x,y_2)$ there exists $w_1'\in G(x,y_1)$ so that: $|w_1'-w_2'|_{L^1(I,R^n)}\leq h(|y_1-y_2|_{L^1(I,R^n)})$ which implies (7) . Using the same method we can prove that for every $y\in A$, the mapping $x\mapsto G(x,y)$ $(x\in S(I,R^n))$ is continuous .

Since T is a compact mapping, all the conditions of Theorem 1 are satisfied .

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Hence, there exists $g \in A$ such that $g \in G(Tg,g)$ which means that g is a solution of functional equation (1). Functional equations (1) include differential inclusions [1].

Some interesting examples of compact integral operators in the space $S(I,R^n)$ are given in [13].

3. AN EXISTENCE RESULT FOR THE EQUATION $x(t) \in F(t,x)$

In this section we shall use the following notations: $(X, \| \ \|)$ is a uniformly convex Banach space; $I = [0,a]$ $(a>0)$ $C(I,X)$ is the space of all continuous functions from I into X, $R_+ = [0,\infty)$, $x_o \in C(I,X)$, $u_o \in C(I,R_+)$; $V(u_o) = \{ u \mid u \in C(I,R)$, $0 \leq u(t) \leq cu_o(t)$, for some $c>0$ and every $t \in I\}$; $D(x_o,u_o) = \{x \mid x \in C(I,X)$, $\|x(t)-x_o(t)\| \leq cu_o(t)$, for some $c>0$ and every $t \in I\}$ and $F : I \times C(I,X) \to CCB(X)$.

Similarly as in [15] let us introduce the following conditions (i)-(iv):

(i) For each $x \in C(I,X)$ the mapping $F(\cdot,x): I \to CCB(X)$ is continuous.

(ii) $Q: V(u_o) \to V(u_o)$ is a nondecreasing operator such that:

$$D(F(t,x),F(t,y)) \leq Q(\|x-y\|)(t)$$

for every $x,y \in D(x_o,u_o)$ and every $t \in I$.

(iii) There exists a $p>0$ such that:

$$D(F(t,x_o),\{x_o(t)\}) \leq p\, u_o(t) \quad \text{for every } t \in I.$$

(iv) There exists a function $h: R_+ \to R_+$ which satisfies conditions (A_1)-(A_4) and:

$$Q(su_o)(t) \leq h(s)u_o(t), \quad t \in I, \text{ for every } s>0.$$

THEOREM 5. *Let all the conditions (i)-(iv) be satisfied. Then there exists $x \in C(I,X)$ so that:*

$$x(t) \in F(t,x), \quad t \in I.$$

Proof: For every $x \in C(I,X)$ let:

$$G(x) = \{f \mid f \in C(I,X), f(t) \in F(t,x), t \in I\}.$$

From the continuity of the mapping $t \mapsto F(t,x)$, for every $x \in C(I,X)$ since $F(t,x) \in CCB(X)$, it follows by Michael's selection theorem that $G(x) \neq \emptyset$, for every $x \in C(I,X)$. Let for every $x,y \in D(x_o,u_o)$:

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$$d_1(x,y) = \sup_{t\in I_o} \frac{\|x(t)-y(t)\|}{u_o(t)}$$

where $I_o=\{t \mid t\in I, u_o(t)\neq \emptyset\}$.

Then $(D(x_o,u_o),d_1)$ is a complete metric space . It is obvious that $G(x)$ is closed in $D(x_o,u_o)$,for every $x\in D(x_o,u_o)$.

Let $x\in D(x_o,u_o)$ and $f\in G(x)$. Since $t\mapsto F(t,x_o)$ is a contin. mapping from I into CB(X) and $(X,\|\ \|)$ is a uniformly convex Bana space from Lemma 5.2 [2] it follows that there exists $g\in C(I,X)$ so that $g(t)\in F(t,x_o)$,for every $t\in I$ and :

$$\|f(t)-g(t)\| = \inf_{z\in F(t,x_o)} \|f(t)-z\| \quad .$$

Then we have

$$\|f(t)-x_o(t)\| \leq \|f(t)-g(t)\| + \|g(t)-x_o(t)\| \leq D(F(t,x),F(t,x_o)) + \|g(t)-x_o(t)\| \leq Q(\|x-x_o\|)(t) + D(F(t,x_o),\{x_o(t)\}) \leq Q(cu_o)(t)+pu_o(t) \leq (h(c)+p)u_o(t).$$

which implies that :

$$d_1(f,x_o) \leq h(c)+p \quad , f\in D(x_o,u_o) \ .$$

Hence $G(x)\in CB(D(x_o,u_o))$.

Let us prove that:

(8) $\overline{D}(G(x),G(y)) \leq h(d_1(x,y))$,for every $x,y\in D(x_o,u_o)$

where $\overline{D}$ is the Hausdorff metric induced by the metric d_1 .

In order to prove (8) we shall show that for every $f\in G(x)$ there exists $f_1\in G(y)$ so that $d_1(f,f_1)\leq h(d_1(x,y))$. If $f\in G(x)$ the

$$\|f(t)-f_1(t)\| = \inf_{z\in F(t,y)} \|f(t)-z\| = D(\{f(t)\},F(t,y)) \leq D(F(t,x),F(t,y)) \leq Q(\|x-y\|)(t) \text{ for some } f_1\in G(y) \quad [2] \ .$$

From $\|x(t)-y(t)\| \leq u_o(t)d_1(x,y)$ $(t\in I)$ it follows that

(9) $$\|f(t)-f_1(t)\| \leq h(d_1(x,y))u_o(t) , t\in I$$

and (9) implies that:

(10) $$d_1(f,f_1) \leq h(d_1(x,y))$$

Similarly ,we can prove that for every $u_1\in G(y)$ there exists $u\in G(x)$ so that :

(11) $$d_1(u,u_1) \leq h(d_1(x,y))$$

Hence,(10) and (11) implies that (8) holds .

Applying Theorem B we conclude that there exists $x\in D(x_o,u_o)$ so that $x\in G(x)$ which means that $x(t)\in F(t,x)$,$t\in I$

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Using Theorem 5 ,similarly as in [15] ,some sufficient conditions for the existence of a solution of functional equations $x(t)\in F(t,x(r(t))$ and integro-functional equations

$$x(t)\in f(t,\int_0^{s(t)} g(t,u,x(u))du,x(r(t))) \ ,t\in I$$

can be given .Here,$r:I\to I$ and $s:I\to I$ are continuous mappings.

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REZIME

NEKE TEOREME O NEPOKRETNOJ TAČKI U BANAHOVIM PROSTORIMA I NJIHOVA PRIMENA

U ovom radu je dobijeno višeznačno uopštenje poznate teoreme Melvina o nepokretnoj tački [16].

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COMMON FIXED POINTS OF TWO PAIRS OF WEAKLY COMMUTING MAPPINGS

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ABSTRACT

A common fixed point theorem is proved involving two pairs of weakly commuting mappings on a metric space (X,d) satisfying
$d(Sx,Ty) \leq g(d(Ix,Jy),d(Ix,Sx),d(Jy,Ty))$, for all x,y in X where $g : [0,\infty)^3 \to [0,\infty)$ satisfies (i) $g(1,1,1) = h < 1$ and (ii) if $u,v \geq 0$ and either $u \leq g(u,v,v)$ or $u \leq g(v,u,v)$ or $u \leq g(v,v,u)$, then $u \leq hv$.

1. THE FIXED POINT THEOREM

In the following, see [6], we defined two mappings S and I of a metric space (X,d) into itself to be weakly commuting if

$$d(SIx,ISx) \leq d(Ix,Sx)$$

for all x in X. It is clear that two commuting mappings weakly commute but two weakly commuting mappings do not necessarily

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commute as is shown in Example 1 of [6].

The following theorem was proved in [3].

Theorem 1. *Let* S *and* I *be commuting mappings and let* T *and* J *be commuting mappings of a complete metric space* (X,d) *into itself satisfying the inequality*

$$(1) \qquad d(Sx,Ty) \leq c \cdot \max\{d(Ix,Jy), d(Ix,Sx), d(Jy,Ty)\}$$

for all x, y *in* X, *where* $0 \leq c < 1$. *If the range of* I *contains the range of* T *and the range of* J *contains the range of* S *and if one of* S, T, I *and* J *is continuous, then* S, T, I *and* J *have a unique common fixed point* z. *Further,* z *is the unique common fixed point of* S *and* I *and of* T *and* J.

Following Delbosco [1], we consider the set S of all real continuous functions $g : [0,\infty)^3 \to [0,\infty)$ satisfying the following properties:

(*i*) $g(1,1,1) = h < 1$,

(*ii*) let $u,v \geq 0$ be such that either $u \leq g(u,v,v)$ or $u \leq g(v,u,v)$ or $u \leq g(v,v,u)$. Then $u \leq hv$.

Delbosco [1] proved the following result.

Theorem 2. *Let* S *and* T *be two mappings of a complete metric space* (X,d) *into itself satisfying the inequality*

$$(2) \qquad d(Sx,Ty) \leq g(d(x,y), d(x,Sx), d(y,Ty))$$

for all x, y *in* X, *where* g *is in* S. *Then* S *and* T *have a unique common fixed point.*

We now unify and generalize Theorems 1 and 2 with the following result.

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Theorem 3. *Let* S *and* I *be weakly commuting mappings and let* T *and* J *be weakly commuting mappings of a complete metric space* (X,d) *into itself satisfying the inequality*

$$d(Sx,Ty) \leq g(d(Ix,Jy),d(Ix,Sx),d(Jy,Ty)) \tag{3}$$

for all x,y *in* X, *where* g *is in* S. *If the range of* I *contains the range of* T *and the range of* J *contains the range of* S *and if one of* S, T, I *and* J *is continuous, then* S, T, I *and* J *have a unique common fixed point* z. *Further,* z *is the unique common fixed point of* S *and* I *and of* T *and* J.

Proof. Let $x = x_0$ be an arbitrary point in X and let x_1 be a point such that $Sx_0 = Jx_1$. This can be done since the range of J contains the range of S. Let x_2 be a point such that $Tx_1 = Ix_2$. This can be done since the range of I contains the range of T. In general, we can choose x_{2n}, x_{2n+1} and x_{2n+2} such that $Sx_{2n} = Jx_{2n+1}$ and $Tx_{2n+1} = Ix_{2n+2}$ for $n = 0,1,2,\ldots$.

Using inequality (3), we have

$$d(Sx_{2n},Tx_{2n+1}) \leq g(d(Ix_{2n},Jx_{2n+1}),d(Ix_{2n},Sx_{2n}),d(Jx_{2n+1},Tx_{2n+1}))$$

$$\leq g(d(Tx_{2n-1},Sx_{2n}),d(Tx_{2n-1},Sx_{2n}),d(Sx_{2n},Tx_{2n+1})),$$

which implies by property (ii)

$$d(Sx_{2n},Tx_{2n+1}) \leq h \cdot d(Tx_{2n-1},Sx_{2n}).$$

Similarly,

$$d(Tx_{2n-1},Sx_{2n}) \leq h \cdot d(Sx_{2n-2},Tx_{2n-1})$$

and so

$$d(Sx_{2n},Tx_{2n+1}) \leq h \cdot d(Tx_{2n-1},Sx_{2n}) \leq h^{2n} \cdot d(Sx_0,Tx_1)$$

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for $n = 1,2,\ldots$. Since $h < 1$, we have that the sequence

$$(4) \qquad \{Sx_0, Tx_1, Sx_2, \ldots, Tx_{2n-1}, Sx_{2n}, Tx_{2n+1}, \ldots\}$$

is a Cauchy sequence in the complete metric space (X,d) and so has a limit z in X. Hence the sequences

$$\{Sx_{2n}\} = \{Jx_{2n+1}\} \quad \text{and} \quad \{Tx_{2n-1}\} = \{Ix_{2n}\},$$

which are subsequences of (4), converge to the point z.

Let us now suppose that the mapping I is continuous, so that the sequences $\{I^2x_{2n}\}$ and $\{ISx_{2n}\}$ converge to the point Iz. Since S and I weakly commute, we have

$$d(SIx_{2n}, ISx_{2n}) \leq d(Ix_{2n}, Sx_{2n})$$

and so the sequence $\{SIx_{2n}\}$ also converges to the point Iz. We now have

$$d(SIx_{2n}, Tx_{2n+1}) \leq g(d(I^2x_{2n}, Jx_{2n+1}), d(I^2x_{2n}, SIx_{2n}), d(Jx_{2n+1}, Tx_{2n+1})).$$

Letting n tend to infinity and since g is continuous, we have

$$d(Iz,z) \leq g(d(Iz,z),0,0) .$$

This implies by property (ii)

$$d(Iz,z) \leq 0$$

and so

$$Iz = z.$$

Further

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$$d(Sz,Tx_{2n+1}) \leq g(d(Iz,Jx_{2n+1}),d(Iz,Sz),d(Jx_{2n+1},Tx_{2n+1}))$$

and letting n tend to infinity, we have

$$d(Sz,z) \leq g(0,d(z,Sz),0)$$

which implies by property (ii)

$$Sz = z.$$

This means that z is in the range of S and since the range of J contains the range of S, there exists a point z' in X such that $Jz' = z$. Thus

$$d(z,Tz') = d(Sz,Tz') \leq g(d(Iz,Jz'),d(Iz,Sz),d(Jz',Tz'))$$

$$= g(0,0,d(z,Tz')),$$

which implies by property (ii)

$$Tz' = z.$$

Since T and J weakly commute, we have

$$d(Tz,Jz) = d(TJz',JTz') \leq d(Jz',Tz') = d(z,z) = 0.$$

Thus $Tz = Jz$ and so

$$d(z,Tz) = d(Sz,Tz) \leq g(d(Iz,Jz),d(Iz,Sz),d(Jz,Tz))$$

$$= g(d(z,Tz),0,0),$$

which implies by property (ii)

$$z = Tz = Jz.$$

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We have therefore proved that z is a common fixed point of S, T, I and J.

If the mapping J is continuous instead of I, then the proof that z is again a common fixed point of S, T, I and J is of course similar.

Now let us suppose that the mapping S is continuous, so that the sequence $\{S^2x_{2n}\}$ and $\{SIx_{2n}\}$ converge to the point Sz. Since S and I weakly commute, it follows as above that the sequence $\{ISx_{2n}\}$ also converges to the point Sz. Thus

$$d(S^2x_{2n},Tx_{2n+1}) \leq g(d(ISx_{2n},Jx_{2n+1}),d(ISx_{2n},S^2x_{2n}),$$
$$d(Jx_{2n+1},Tx_{2n+1})).$$

Letting n tend to infinity, we have

$$d(Sz,z) \leq g(d(Sz,z),0,0)$$

and so

$$Sz = z$$

by property (ii). Once again there exists a point z' in X such that $Jz' = z$. Thus

$$d(S^2x_{2n},Tz') \leq g(d(ISx_{2n},Jz'),d(ISx_{2n},S^2x_{2n}),d(Jz',Tz')).$$

Letting n tend to infinity, it follows that

$$d(z,Tz') \leq g(0,0,d(z,Tz'))$$

and so

$$Tz' = z$$

by property (ii). Since T and J weakly commute, it again

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follows as above that

$$Tz = Jz.$$

Further

$$d(Sx_{2n},Tz) \leq g(d(Ix_{2n},Jz),d(Ix_{2n},Sx_{2n}),d(Jz,Tz)).$$

Letting n tend to infinity, it follows that

$$d(z,Tz) \leq g(d(z,Tz),0,0)$$

and so

$$Tz = z = Jz$$

by property (ii). The point z is therefore in the range of T and since the range of I contains the range of T, there exists a point z'' in X such that $Iz'' = z$. Thus

$$d(Sz'',z) = d(Sz'',Tz) \leq g(d(Iz'',Jz),d(Iz'',Sz''),d(Jz,Tz))$$

$$= g(0,d(z,Sz''),0)$$

and so

$$Sz'' = z$$

by property (ii). Since S and I weakly commute, we have

$$d(Sz,Iz) = d(SIz'',ISz'') \leq d(Iz'',Sz'') = d(z,z) = 0.$$

Thus

$$Sz = Iz = z .$$

We have therefore proved once again that z is a com-

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mon fixed point of S, T, I and J.

If the mapping T is continuous instead of S, then the proof that z is again a common fixed point of S, T, I and J is similar.

Now let w be a second common fixed point of S and I. Using inequality (3), we have

$$d(w,z) = d(Sw,Tz) \leq g(d(Iw,Jz),d(Iw,Sw),d(Jz,Tz))$$

$$= g(d(w,z),0,0)$$

and it follows, from property (ii), that w = z. Then z is the unique common fixed point of S and I. Similarly, it is proved that z is the unique common fixed point of T and J. This completes the proof of the theorem.

Remark 1. Assuming $g(t_1,t_2,t_3) = c \cdot max\{t_1,t_2,t_3\}$ for any $t_1,t_2,t_3 \geq 0$, it is easily seen that g is in S. Then Theorem 1 follows from Theorem 3.

The following example shows that Theorem 3 is a stronger result than Theorem 1.

Example 1. Let $X = \{1,2,3,4\}$ be a finite set with metric d defined by

$$d(1,2) = 3, \quad d(1,3) = 7.9, \quad d(1,4) = 7.95,$$
$$d(2,3) = 6, \quad d(2,4) = 8, \quad d(3,4) = 12.$$

Let I = J be the identity on X and define S and T by

$$S1 = S3 = S4 = 3, \quad S2 = 4, \quad T1 = T3 = T2 = 3, \quad T4 = 1.$$

A routine calculation shows that Theorem 3 is satisfied if one assumes that

$$g(t_1,t_2,t_3) = hk^{(t_1-t_2)^2(t_2-t_3)^2(t_3-t_1)^2} \cdot max\{t_1,t_2,t_3\},$$

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where h and k are real numbers such that

$$\frac{7.95}{8} \leq h < 1, \quad k > 1.$$

However, inequality (1) does not hold since we have for $x = 2$ and $y = 3$

$$d(S2,T3) = d(4,3) = 12 > 8 = max\{6,8,0\}$$

$$= max\{ d(2,3),d(2,S2),d(3,T3)\}.$$

Remark 2. If I = J is the identity on X, Theorem 3 becomes Theorem 2.

Remark 3. We refer to the examples of [3], where it is shown that the weak commutativity (see [7]) of T and J, the range of I contains the range of T and the continuity of one of the mappings S, T, I and J are necessary conditions in Theorem 1 and therefore also in Theorem 3.

2. A COMPARISON

The authors of [4], generalizing Theorem 1, considered the family $\mathcal{F}$ of all real functions $f : [0,\infty) \to [0,\infty)$ such that f is increasing, continuous from the right and $f(t) < t$ for any $t > 0$. In particular, they established, under suitable assumptions for the mappings S, T, I and J, a similar result to Theorem 3 using the following inequality:

$$(5) \quad d(Sx,Ty) \leq f(max\{d(Ix,Jy),d(Ix,Sx),d(Jy,Ty),d(Ix,Ty),d(Jy,Sx)\})$$

for all x,y in X, where f is in $\mathcal{F}$. Of course, this result extends the results of [2] and [7], where the authors considered analogous contractive conditions by using a function f in $\mathcal{F}$. As pointed out by Delbosco [1], a function g in S generally does

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not belong to $\mathcal{F}$ and vice versa, i.e. inequality (3) and inequality (5) are two different generalizations of inequality (1).
We illustrate this with suitable examples.

Example 2. Let X, S, T, I, J and d be as in Example 1. We already know that inequality (3) holds. We now show that inequality (5) is not satisfied. Indeed, we have for $x = 2$, $y = 3$ and for any f in $\mathcal{F}$

$$d(S2,T3) = d(4,3) = 12 > f(12) = f(\max\{6,8,0,6,12\})$$
$$= f(\max\{d(I2,J3),d(S2,J3),d(I2,S2),d(J3,S3),d(I2,T3)\}) .$$

Adopting the same technical proof of Theorem 3, it is not difficult to prove the following result.

Theorem 4. *Let* S *and* I *be weakly commuting mappings and let* T *and* J *be weakly commuting mappings of a complete metric space* (X,d) *into itself satisfying the inequality*

$$(6)\quad d(Sx,Ty) \le \max\{cd(Ix,Jy),cd(Ix,Sx),cd(Jy,Ty),ad(Ix,Ty) + bd(Jy,Sx)\}$$

for all x,y *in* X, *where* a, b, c *are real numbers such that* $0 \le c < 1$, $0 \le a + b < 1$ *and*

$$(7)\qquad c\cdot\max\{a/(1-a),b/(1-b)\} < 1.$$

If the range of I *contains the range of* T *and the range of* J *contains the range of* S *and if one of* S, T, I *and* J *is continuous, then* S, T, I *and* J *have a unique common fixed point* z. *Further,* z *is the unique common fixed point of* S *and* I *and of* T *and* J.

Proof. Let $x = x_0$ be an arbitrary point in X and, as in the proof of Theorem 3, we consider the sequence (4)

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such that $Sx_{2n} = Jx_{2n+1}$ and $Tx_{2n+1} = Ix_{2n+2}$ for $n = 0,1,2,\ldots$.

Using inequality (6), we have

$$d(Sx_{2n},Tx_{2n+1}) \leq max\{cd(Ix_{2n},Jx_{2n+1}),cd(Ix_{2n},Sx_{2n}),$$

$$cd(Jx_{2n+1},Tx_{2n+1}),ad(Ix_{2n},Tx_{2n+1}) + bd(Jx_{2n+1},Sx_{2n})\}$$

$$\leq max\{cd(Tx_{2n-1},Sx_{2n}),cd(Sx_{2n},Tx_{2n+1}),$$

$$a[d(Tx_{2n-1},Sx_{2n}) + d(Sx_{2n},Tx_{2n+1})]\}$$

which implies that

$$d(Sx_{2n},Tx_{2n+1}) \leq max\{c,a/(1-a)\}\cdot d(Tx_{2n-1},Sx_{2n}).$$

Similarly

$$d(Tx_{2n-1},Sx_{2n}) \leq max\{c,b/(1-b)\}\cdot d(Sx_{2n-2},Tx_{2n-1})$$

and so

$$(8) \qquad d(Sx_{2n},Tx_{2n+1}) \leq \alpha^n\cdot d(Sx_0,Tx_1)$$

and

$$(9) \qquad d(Tx_{2n-1},Sx_{2n}) \leq \alpha^{n-1}\, d(Tx_1,Sx_2),$$

where

$$\alpha = max\{c,a/(1-a)\}\cdot max\{c,b/(1-b)\}.$$

It is easily seen that $0 \leq \alpha < 1$. This implies, by (8) and (9), that the sequence (4) is a Cauchy sequence in the complete metric space X and so has a limit z. From now on, the proof is similar to the proof of Theorem 3 and we therefore omit it.

Remark 4. If I = J is the identity on X, Theorem 4 becomes Theorem 6 of [5].

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Remark 5. Assuming $f(t) = \max\{ct, at + bt\}$ for any $t > 0$ with $0 \leq c < 1$ and $0 \leq a + b < 1$, it is immediately seen that f is in $\mathcal{F}$. Theorem 4 can therefore also be obtained as a consequence of Theorem 1 of [4].

Since inequality (6) is a particular case of inequality (5), we now give example where inequality (6) is satisfied but inequality (3) does not hold.

Example 3. Let $X = \{1,2,3,4\}$ be a finite set with metric d defined by

$$d(1,3) = d(1,4) = d(2,3) = d(2,4) = 1,$$

$$d(1,2) = d(3,4) = 2.$$

Let I be the identity on X and define S, T, J on X by

$$S1 = 2, \quad S2 = S3 = 1, \quad S4 = 3,$$

$$T1 = T2 = T3 = T4 = 4,$$

$$J1 = 2, \; J2 = 1, \quad J3 = 3, \quad J4 = 4.$$

All the conditions of Theorem 4 are satisfied since

$$S(X) = \{1,2,4\} \subset X = J(X),$$

$$T(X) = \{4\} \subset X = I(X)$$

and of course S, T, I and J are continuous. Further, it is easily seen that S and T commute with I and J respectively and inequalities (6) and (7) hold with $a = 1/2$, $b = 0$ and $c = 1/2$. Inequality (3) is not satisfied. Indeed, we have for $x = 3$, $y = 1$ and any g in S:

$$d(S3,T1) = d(1,4) = 1 > g(1,1,1)$$

$$= g(d(I3,J1), d(I3,S3), d(J1,T1)).$$

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We now observe that inequality (6) implies

$$d(Sx,Ty) \leq max\{cd(Ix,Jy), cd(Ix,Sx), cd(Jy,Ty), (a+b)\cdot max\{d(Ix,Ty), d(Jy,Sx)\}\}$$

for all x,y in X and so

$$(10) \qquad d(Sx,Ty) \leq \beta\cdot max\{d(Ix,Jy), d(Ix,Sx), d(Jy,Ty), d(Ix,Ty), d(Jy,Sx)\}$$

for all x,y in X, where $0 \leq \beta = max\{c,a+b\} < 1$.

We now show that Theorem 4 fails if we assume the more general inequality (10) instead of inequality (6) when $1/2 \leq \beta < 1$. To see this, we give the following example mainly inspired by Example 6 of [5].

Example 4. $X = \{1,2,3,4\}$ be a finite set with metric d defined by

$$d(1,2) = d(3,4) = 2, \quad d(1,3) = d(2,4) = 1,$$

$$d(1,4) = d(2,3) = 3/2.$$

Let I be the identity on X and define S, T, J on X by

$$S1 = S4 = 2, \quad S2 = S3 = 1,$$

$$T1 = T3 = 4, \quad T2 = T4 = 3,$$

$$J1 = 3, \quad J2 = 4, \quad J3 = 1, \quad J4 = 2.$$

We have

$$S(X) = \{1,2\} \subset X = J(X)$$

$$T(X) = \{3,4\} \subset X = I(X)$$

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and of course S, T, I and J are continuous. Further,

$$d(TJ1,JT1) = d(T3,J4) = d(4,2) = 1 < 2 = d(3,4) = d(J1,T1),$$

$$d(TJ2,JT2) = d(T4,J3) = d(3,1) = 1 < 2 = d(4,3) = d(J2,T2),$$

$$d(TJ3,JT3) = d(T1,J4) = d(4,2) = 1 < 3/2 = d(1,4) = d(J3,T3),$$

$$d(TJ4,JT4) = d(T2,J3) = d(3,1) = 1 < 3/2 = d(2,3) = d(J4,T4).$$

Thus T weakly commutes with J and S commutes with I. It is easily seen that inequality (10) is satisfied with $1/2 \leq \beta < 1$, but S, T, I and J do not have common fixed points.

We conclude observing that if $0 \leq \beta < 1/2$, then inequality (10) implies

$$d(Sx,Ty) \leq \max\{\beta d(Ix,Jy), \beta d(Ix,Sx), \beta d(Jy,Ty), \beta d(Ix,Ty) + \beta d(Jy,Sx)\}$$

for all x,y in X. This inequality is formally analogous to (6) by putting $\beta = a = b = c$. Therefore Theorem 4 remains valid if inequality (6) is replaced by the equivalent inequality (10), provided that $0 \leq \beta < 1/2$. In this case we note that inequality (7) is also satisfied, since $\beta < 1/2$ implies that

$$\beta \frac{\beta}{1-\beta} < \frac{1}{2} < 1.$$

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REZIME

ZAJEDNIČKE NEPOKRETNE TAČKE DVA PARA SLABO KOMUTATIVNIH PRESLIKAVANJA

Dokazana je teorema o zajedničkoj nepokretnoj tački za dva para slabo komutativnih preslikavanja definisanih nad metričkim prostorom (X,d), koji zadovoljavaju:

$$d(Sx,Ty) \leq g(d(Ix,Jy),d(Ix,Sx),d(Jy,Ty)), \text{ za sve } x,y \text{ u } X$$

gde $g: [0,\infty)^3 \to [0,\infty)$ zadovoljava:

(i) $g(1,1,1) = h < 1$

(ii) ako je $u,v \geq 0$ i ili $u \leq g(u,v,v)$ ili $u \leq g(v,u,v)$ ili $u \leq g(v,v,u)$ tada je $u \leq hv$.

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CARATHEODORY SELECTIONS, RANDOM FIXED-POINT THEOREMS AND EXISTENCE OF EQUILIBRIA

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ABSTRACT

In this paper a Caratheodory-type selection theorem is proved. As an application, a random fixed point theorem is obtained. A result on the existence of equilibria in abstract economies with a measure space of agents and with an infinite-dimensional strategy space is also included.

1. INTRODUCTION

In a series of papers, Kim-Prikry-Yannelis [2, 3, 4] have proved a Caratheodory-type selection theorem and used it to prove general theorems on the existence of random fixed-points and the existence of equilibria in abstract economies with a measure space of agents and with an infinite-dimensional strategy space.

The object of this paper is to prove similar results. The paper is organized as follows. Section 2 contains definitions and preliminary material. In section 3, a Caratheodory-type selection theorem is proved. In section 4, a random fixed-point theorem is proved. In section 5, a theorem on the

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existence of equilibrium in abstract economies is proved by using the Caratheodory-type selection theorem established in section 3.

2. PRELIMINARIES

Let X,Y be two topological spaces. The *graph* of the correspondence $\phi : X \to 2^Y$ is denoted by $G_\phi = \{(x,y) \in X \times Y : y \in \phi(x)\}$. The correspondence $\phi : X \to 2^Y$ is said to have a *closed graph* if the set G_ϕ is closed in $X \times Y$. A correspondence $\phi: X \to 2^Y$ is said to have *open lower sections* if for each $y \in Y$ the set $\phi^{-1}(y) = \{x \in X : y \in \phi(x)\}$ is open in X. An *open cover* of a topological space X is a collection $U = \{u_a : a \in A\}$ of open subsets of X whose union is X, i.e. $\bigcup_{a \in A} u_a = X$. An open cover $U = \{u_a : a \in A\}$ is *locally finite* if every $x \in X$ has a neighborhood intersecting only finitely many $u \in U$. Let (T,τ,μ) be a complete finite measure space, i.e. μ is a realvalued, non-negative, countably additive measure defined in a complete σ-field τ of subsets of T such that $\mu(T) < \infty$. The correspondence $\phi: T \to 2^X$ is a said to have a *measurable graph* if $G_\phi \in \tau \times B(X)$, where B(X) denotes Borel σ-algebra on X and $\times$ denotes σ-product field.

Let now X be a Banach space. $L_1(\mu,X)$ denotes the space of equivalence classes of X-valued Bochner integrable functions $f : T \to X$ normed by

$$\|f\| = \int_T \|f(t)\| d\mu(t).$$

A correspondence $\phi : T \to 2^X$ is said to be *integrably bounded* if there exists a map $g \in L_1(\mu)$ such that for almost all $t \in T$, $\sup\{\|x\| : x \in \phi(t)\} \leq g(t)$. A Banach space X has the *Radon-Nikodym property* with respect to (T,τ,μ) if for each μ-continuous vector measure $G : \tau \to X$ of bounded variation there exists $g \in L_1(\mu,X)$ such that $G(E) = \int_E g d\mu$ for all $E \in \tau$.

Let X be a topological space and Y be a linear topological space. Let $\phi : X \to 2^Y$ be a nonempty valued correspondence. A function $f : X \to Y$ is said to be a *continuous se-*

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lection from ϕ if $f(x) \in \phi(x)$ for all $x \in X$, and f is continuous. Let T be an arbitrary measure space. Let $\psi : T \to 2^Y$ be a nonempty valued correspondence. A function $f : T \to Y$ is said to be a *measurable selection* from ψ if $f(t) \in \psi(t)$ for all $t \in T$, and f is measurable. Let Z be a topological space and $\phi : T \times Z \to 2^Y$ be a nonempty valued correspondence. A function $f : T \times Z \to Y$ is said to be a *Caratheodory-type selection* from ϕ if $f(t,z) \in \phi(t,z)$ for all $(t,z) \in T \times Z$ and $f(\cdot,z)$ is measurable for all $z \in Z$ and $f(t,\cdot)$ is continuous for all $t \in T$.

3. SELECTION THEOREM

The following selection theorem is due to Kim-Prikry-Yannelis [2].

Theorem 1 *Let (T,τ,μ) be a complete measure space, Y be a complete, metrizable and separable Hausdorff linear topological space and Z be a complete metrizable and separable topological space. Let $X : T \to 2^Y$ be a correspondence with measurable graph, and $\phi : T \times Z \to 2^Y$ be a convex valued correspondence (possibly empty) with measurable graph, such that:*

(i) *for each $t \in T$, $\phi(t,x) \subseteq X(t)$ for all $x \in Z$.*

(ii) *for each t, $\phi(t,\cdot)$ has open lower sections in Z, i.e. for each $t \in T$, and each $y \in Y$, $\phi_t^{-1}(y) = \{x \in Z : y \in \phi(t,x)\}$ is open in Z.*

(iii) *for each $(t,x) \in T \times Z$, such that $\phi(t,x) \neq \emptyset$, $\phi(t,x)$ has a nonempty interior in $X(t)$.*

Let $U = \{(t,x) \in T \times Z : \phi(t,x) \neq \emptyset\}$ and for each $x \in Z$, $U_x = \{t \in T : (t,x) \in U\}$ and for each $t \in T$, $U^t = \{x \in Z : (t,x) \in U\}$. Then for each $x \in Z$, U_x is a measurable set in T and there exists a Caratheodory-type selection from $\phi|_U$, i.e., there exists a function $f : U \to Y$ such that $f(t,x) \in \phi(t,x)$ for all $(t,x) \in U$ and for each $x \in Z$, $f(\cdot,x)$ is measurable on U_x and for each $t \in T$, $f(t,\cdot)$ is continuous on U^t. Moreover, $f(\cdot,\cdot)$ is jointly measurable.

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We now prove the following selection theorem.

Theorem 2 *Let T,Y and Z be as in Theorem 1. Suppose that* $\phi : T \times Z \to 2^Y$ *is a convex-valued correspondence (possibly empty) such that the following conditions hold:*

(i)′ if S is a countable dense subset of Y, then for each $t \in T$, *and* $x \in U^t = \{z \in Z : (t,z) \in \in U\}$, *there exists* $y \in S$ *such that* $x \in int\phi_t^{-1}(y)$ *where* $\phi_t^{-1}(y) = \{z \in Z : y \in \phi(t,z)$

(ii)′ for each $t \in T$, *and* $y \in Y$ *the correspondence* $B : T \to 2^Z$ *given by* $B(t) = int\{z \in Z:: y \in \phi(t,z)$ *has a measurable graph.*

Then the conclusion of Theorem 1 holds, i.e. ϕ *has a Caratheodory-type selection f, that is jointly measurable.*

Remark The Theorem 2 is different from the Theorem 1 in the following ways:

1. The conditions (i) and (iii) of Theorem 1 are altogether removed in Theorem 2.
2. The condition (i)′ of Theorem 2 is weaker than the conditions (ii) and (iii) of Theorem 1. To see this suppose that conditions (ii) and (iii) of Theorem 1 hold. Let S be a countable dense subset of Y and $t \in T$ with $x \in U^t$.

Now $x \in U^t$ implies that $\phi(t,x) \neq \phi$. Hence, the condition (iii) implies that there exists $y \in \phi(t,x) \cap S$, since S is dense. Now $y \in \phi(t,x)$ implies $x \in \phi_t^{-1}(y)$. Condition (ii) implies that $\phi_t^{-1}(y)$ is open. Hence, $x \in int\phi_t^{-1}(y) = \phi_t^{-1}(y)$ and condition (i)′ of theorem holds. Thus conditions (ii) and (iii) of Theorem 1 imply condition (i)′ of Theorem 2.

3. The measurability of $\phi : T \times Z \to 2^Y$ in Theorem 1 has been replaced by the condition (ii)′. We do not know if the two measurability assumptions are related. It seems that they are not comparable.

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Proof The general line of argument is the same as in Kim-Prikry-Yannelis [2]. As in [2], U_x is a measurable set. Since Y is separable there exists a countable dense subset $\{y_1, y_2, \ldots,\}$ of Y. For each $n = 1,2,\ldots$, define a function $f_n : T \to Y$ by $f_n(t) = y_n$ for all $t \in T$. Each f_n is clearly measurable.

Now for each $t \in T$ and $x \in U^t$, assumption (i)′ implies that there exists $f_n(t)$ for some n, such that $x \in int\phi_t^{-1}(f_n(t)) = B_n(t)$. This implies that the open sets $\{B_n(t) : n = 1,2,\ldots\}$ form an open cover of the set U^t, and $B_n(t) \subseteq \{z \in Z : f_n(t) \in \phi(t,z)\}$. Assumption (ii)′ implies that $B_n(\cdot)$ has a measurable graph.

For each $m = 1,2,\ldots$ define the operator $(\ )_m$ by $(W)_m = \{w \in W : dist(w, Z\backslash W) \geq \frac{1}{2^m}\}$. For $n = 1,2,\ldots$ and $t \in T$, let $C_n(t) = B_n(t)\backslash \bigcup_{k=1}^{n-1} (B_k(t))_n$.
Then $C_n(t)$ is open in Z and it can be easily checked that $\{C_n(t) : n = 1,2,\ldots\}$ is a locally finite open cover of the set $\{x \in Z : (t,x) \in U\} = U^t$.

Since $B_n(\cdot)$ has a measurable graph, it follows that $C_n(\cdot)$ has a measurable graph by Lemmas 4.6 and 4.8 of Kim-Prikry-Yannelis [2].

Define, for $n = 1,2,\ldots$, $g_n(t,x) = \dfrac{dist(x, Z\backslash C_n(t))}{\sum_{k=1}^{\infty} dist(x, Z\backslash C_k(t))}$.

This is a partition of unity subordinated to the open cover $\{C_n(t) : n = 1,2,\ldots\}$.
Define $f : U \to Y$ by $f(t,x) = \sum_{n=1}^{\infty} g_n(t,x) f_n(t)$. Since $\{C_n(t) : n = 1,2,\ldots\}$ is locally finite, each x has a neighborhood N_x which intersects only finitely many $C_n(t)$. Hence, $f(t,\cdot)$ is a finite sum of continuous functions on N_x and it is therefore continuous. Furthermore, for any n such that $g_n(t,x) > 0$, $x \in C_n(t) \subseteq B_n(t) \subseteq \{z \in Z : f_n(t) \in \phi(t,z)\}$, i.e., $f_n(t) \in \phi(t,x)$. So $f(t,x)$ is a convex combination of elements $f_n(t)$ from the convex set $\phi(t,x)$. Consequently, $f(t,x) \in \phi(t,x)$ for all $(t,x) \in U$. Since $C_n(\cdot)$ has a measurable graph, $dist(x, Z \backslash C_n(\cdot))$ is a measurable function by Lemmas 4.6 and 4.7 of

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Kim-Prikry-Yannelis [2]. Therefore for each n and x, $g_n(\cdot,x)$ is a measurable function. Since for each n, $f_n(\cdot)$ is a measurable function, it follows that $f(\cdot,x)$ is measurable for each x, i.e., $f(t,x)$ is a Caratheodory-type selection from $\phi|_U$.

The joint measurability of f follows from Lemma 4.12 of Kim-Prikry-Yannelis [4].

4. A RANDOM FIXED-POINT THEOREM

Let T be any measure space and X be a nonempty subset of any linear topological space. Let ϕ be a correspondence from $T \times X$ into 2^X. The correspondence ϕ is said to have a *random fixed point* if there exists a measurable function $x : T \to X$ such that $x(t) \in \phi(t,x(t))$ for almost all $t \in T$.

Below we prove a random fixed-point theorem.

Theorem 3. *Let (T,τ,μ) be a complete finite measure space, and K be a nonempty compact convex subset of a separable, complete and metrizable locally convex linear topological space E. Let $\phi : T \times K \to 2^K$ be a non-empty convex valued map such that*

(i) if S is a countable dense subset of E, then for each $t \in T$, $x \in U^t = \{z \in K : \phi(t,z) \neq \phi\}$ there exists $y \in S$ such that $x \in int\ \phi_t^{-1}(y)$ where $\phi_t^{-1}(y) = \{z \in K : y \in \phi(t,z)\}$.

(ii) for each $t \in T$ and $y \in Y$ the map $B : T \to 2^K$ given by $B(t) = int\{z \in K : y \in \phi(t,z)\}$ has a measurable graph.

Then ϕ has a random fixed-point.

Proof. All the asumptions of Theorem 2 are satisfied for ϕ. Hence, Theorem 2 implies that there exists a jointly measurable Caratheodory-type selection f of ϕ, i.e., $f(t,x) \in \phi(t,x)$ for all $(t,x) \in T \times K$.

Define $F : T \to K$ by $F(t) = \{x \in K : x - f(t,x) = 0\}$.

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$F(t) \neq \phi$ by Tychonoff's fixed point theorem. F has a measurable graph since f is jointly measurable. Hence, Aumann's measurable selection theorem implies that there exists a measurable function $x^* : T \to K$ such that $x^*(t) \in F(t)$ for almost all $t \in T$. Consequently, $x^*(t) = f(t,x^*) \in \phi(t,x^*)$ for almost all $t \in T$, and the theorem is proved.

Remark The theorem proved above is similar to Theorem 3.3 of Kim-Prikry-Yannelis [4]. It should be noted that unlike Kim-Prikry-Yannelis we do not assume that ϕ has closed values or that E is a separable Banach space.

5. EQUILIBRIUM EXISTENCE THEOREM

Let (T,τ,μ) be a finite, positive, complete measure space. Let Y be a separable Banach space whose dual possesses the Radon-Nikodym property. For any correspondence $X : T \to 2^Y$, $L_1(\mu,X)$ will denote the subset of $L_1(\mu,X)$ consisting of those $x \in L_1(\mu,X)$ which satisfy $x(t) \in X(t)$ for almost all t in T.

An *abstract economy* Γ is a quadruple $[(T,\tau,\mu),X,P,A]$, where

(1) (T,τ,μ) is a measure space of agents;

(2) $X : T \to 2^Y$ is a strategy correspondence;

(3) $P : T \times L_1(\mu,X) \to 2^Y$ is a preference correspondence such that $P(t,x) \subseteq X(t)$ for all $(t,x) \in T \times L_1(\mu,X)$;

(4) $A : T \times L_1(\mu,X) \to 2^Y$ is a correspondence such that $A(t,x) \subseteq X(t)$ for all $(t,x) \in T \times L_1(\mu,X)$.

Notice that since P is a mapping from $T \times L_1(\mu,X)$ to 2^Y, we have allowed for interdependent preferences. The interpretation of these preference correspondences is that $y \in P(t,x)$ means that agent t strictly prefers y to x(t) if the given strategies of other agents are fixed. Notice that preferences need not be transitive or complete and therefore need not be representable by utility functions. However, it will be assumed that $x(t) \notin con\ P(t,x)$, the convex hull of P(t,x), for all $x \in L_1(\mu,X)$ and for almost all t in T, which implies that

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$x(t) \notin P(t,x)$ for all $x \in L_1(\mu,X)$ and almost all t in T, i.e., $P(t,\cdot)$ is *irreflexive* for almost all t in T.

An *equilibrium* for Γ is an $x^* \in L_1(\mu,X)$ such that for almost all t in T the following conditions are satisfied:

(i) $x^*(t) \in cl A(t,x^*)$, and

(ii) $P(t,x^*) \cap cl A(t,x^*) = \phi$.

The following equilibrium existence theorem is due to Kim-Prikry-Yannelis [2].

Theorem 4 *Let* $\Gamma = [(T,\tau,\mu),X,P,A]$ *be an abstract economy satisfying the assumptions* (A.1)-(A.4) *where:*

(A.1) (T,τ,μ) *is a finite, positive, complete, separable measure space.*

(A.2) $X : T \to 2^Y$ *is an integrably bounded correspondence with measurable graph such that for all* $t \in T$, $X(t)$ *is a nonempty, convex and weakly compact subset of* Y, *where* Y *is a separable Banach space whose dual possesses the Radon-Nikodym property.*

(A.3) $A : T \times L_1(\mu,X) \to 2^Y$ *is a correspondence such that:*

(a) $\{(t,x,y) \in T \times L_1(\mu,X) \times Y : y \in A(t,x)\} \in \tau \times B_w(L_1(\mu,X)) \times B(Y)$ *where* $B_w(L_1(\mu,X))$ *is the Borel* σ*-algebra for the weak topology on* $L_1(\mu,X)$ *and* $B(Y)$ *is the Borel* σ*-algebra for the norm topology on* Y;

(b) *it has weakly open lower sections, i.e., for each* $t \in T$ *and for each* $y \in Y$, *the set* $A^{-1}(t,y) = \{x \in L_1(\mu,X) : y \in A(t,x)\}$ *is weakly open in* $L_1(\mu,X)$;

(c) *for all* $(t,x) \in T \times L_1(\mu,X)$, $A(t,x)$ *is convex and has a nonempty norm interior in* $X(t)$;

(d) *for each* $t \in T$, *the correspondence* $\bar{A}(t,\cdot) : L_1(\mu,X) \to 2^Y$, *defined by* $\bar{A}(t,x) = cl A(t,x)$

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for all $(t,x) \in T \times L_1(\mu,X)$, *is u.s.c. in the sense that the set* $\{x \in L_1(\mu,X) : \bar{A}(t,x) \subseteq V\}$ *is weakly open in* $L_1(\mu,X)$ *for every norm open subset* V *of* Y.

(A.4) $P : T \times L_1(\mu,X) \to 2^Y$ *is a correspondence such that*

(a) $\{(t,x,y) \in T \times L_1(\mu,X) \times Y : y \in conP(t,x)\} \in$
$\in \tau \times B_w(L_1(\mu,X)) \times B(Y)$

(b) *it has weakly open lower sections, i.e., for each* $t \in T$ *and each* $y \in Y$, $P^{-1}(t,y) = \{x \in$
$\in L_1(\mu,X) : y \in P(t,x)\}$ *is weakly open in* $L_1(\mu,X)$;

(c) *for all* $(t,x) \in T \times L_1(\mu,X)$, $P(t,x)$ *is norm open* $X(t)$;

(d) $x(t) \notin con\ P(t,x)$ for all $x \in L_1(\mu,X)$ *and for almost all* t *in* T.

Then Γ *has an equilibrium.*

We now use Theorem 2 to weaken some of the topological assumptions Theorem 4.

Theorem 5 *Assume that all the conditions of Theorem 3, except (A.3b) and (A.4b), hold. Suppose that in addition, the following conditions hold:*

(i) *if* S *is a countable dense subset of* Y, *then for each* $t \in T$, $x \in L_1(\mu,X)$ *such that* $A(t,x) \cap$
$\cap\ conP(t,x) \neq \phi$, *there exists* $y \in S$ *such that* $x \in int(A_t^{-1}(y) \cap conP_t^{-1}(y))$

(ii) *for each* $t \in T$, *and* $y \in Y$, *the correspondence* $B : T \to 2^{L_1(\mu,X)}$ *given by*
$B(t) = int\{z \in L_1(\mu,X) : y \in A(t,z) \cap conP(t,z)\}$
has a measurable graph.

Then the economy Γ *has an equilibrium.*

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Proof Although the proof is similar to that in [3], we include it for the sake of completeness.
Define $\psi : T \times L_1(\mu,X) \to 2^Y$ by $\psi(t,x) = con\ P(t,x)$ for $(t,x) \in$ $\in T \times L_1(\mu,X)$ and $\phi : T \times L_1(\mu,X) \to 2^Y$ by $\phi(t,x) = A(t,x) \in$ $\in \psi(t,x)$.

Assumptions (i) and (ii) imply that all the conditions of Theorem 2 are satisfied. Hence, Theorem 2 implies that there exists a function $f : U \to Y$ such that $f(t,x) \in \phi(t,x)$ for all $(t,x) \in U$, and for each $x \in L_1(\mu,X)$, $f(\cdot,x)$ is measurable on U_x and for each $t \in T$, $f(t,\cdot)$ is continuous on U^t where $L_1(\mu,X)$ is endowed with the weak topology and Y with the norm topology. Moreover, for each $x \in L_1(\mu,X)$, U_x is a measurable set.
Define $\theta : T \times L_1(\mu,X) \to 2^Y$ by

$$\theta(t,x) = \begin{cases} \{f(t,x)\} \text{ if } (t,x) \in U \\ \\ c\ell A(t,x) \text{ if } (t,x) \notin U. \end{cases}$$

By Lemma 4.2 Kim-Prikry-Yannelis [3] for each $x \in L_1(\mu,X)$, the correspondence $c\ell A(\cdot,x) : T \to 2^Y$ has a measurable graph. Therefore, by Lemma 4.3 of Kim-Prikry-Yannelis for each $x \in L_1(\mu,x)$, $\theta(\cdot,x) : T \to 2^Y$ has a measurable graph. Notice that assumption (i) implies that the set U^t is weakly open in $L_1(\mu,X)$. Consequently, by Lemma 4.5 of Kim-Prikry-Yannelis [3], $\theta(t,\cdot) : L_1(\mu,X) \to 2^Y$ is u.s.c. in the sense that the set $\{x \in L_1(\mu,X) : \theta(t,x) \subseteq V\}$ is weakly open in $L_1(\mu,X)$ for every norm open subset V of Y. Moreover, θ is convex and non-empty valued. Define $F : L_1(\mu,X) \to 2^{L_1(\mu,X)}$ by $F(x) = \{y \in$ $\in L_1(\mu,X)$: for almost all t in T, $y(t) \in \theta(t,x)\}$. Since for each $x \in L_1(\mu,X)$, $\theta(\cdot,x)$ has a measurable graph, F is nonempty valued by Lemma 4.6 of Kim-Prikry-Yannelis [3]. Since θ is convex valued, so is F. By Lemma 4.9 and Remark 4.2 of Kim-Prikry-Yannelis [3] F is weakly u.s.c. Furthermore, since $X(\cdot)$ is integrably bounded and has a measurable graph, $L_1(\mu,X)$ is nonempty as a consequence of Aumann's measurable selection theorem; also since $X(\cdot)$ is convex valued, $L_1(\mu,X)$ is convex. By

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Lemma 4.8.and Remark 4.1 of Kim-Prikry-Yannelis [3] $L_1(\mu,X)$ is weakly compact. Therefore, by the Fan fixed point theorem, there exists $x^* \in L_1(\mu,X)$ such that $x^* \in f(x^*)$, i.e., $x^*(t) \in \in \theta(t,x^*)$ for almost all t in T. We now show that the fixed point is by construction an equilibrium for Γ. Suppose that for a non-null set of agents $S,(t,x^*) \in U$ for all $t \in S$. Then by the definition of θ $x^*(t) = f(t,x^*) \in \phi(t,x^*) \subseteq con P(t,x^*)$ for all $t \in S$, a contradiction to assumption (A.4)d. Therefore, $(t,x^*) \notin U$ for almost all t in T and so for almost all $t \in T$, $x^*(t) \in \in cl A(t,x^*)$ and $\phi(t,x^*) = A(t,x^*) \cap con P(t,x^*) = \phi$. But, $A(t,x^*) \cap con P(t,x^*) = \phi$ implies that $A(t,x) \cap P(t,x^*) = \phi$. Since by assumption (A.3)c, $P(t,x)$ is norm open in $X(t)$ for all $(t,x) \in T \times L_1(\mu,X)$, the fact that $A(t,x^*) \cap P(t,x^*) = \phi$, implies that $cl A(t,x^*) \cap P(t,x^*) = \phi$, i.e., x^* is an equilibrium for Γ. This completes the proof of the main existence theorem.

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REZIME

KARATEODORIJEVA SELEKCIJA, TEOREME O STOHASTIČKOJ NEPOKRETNOJ TAČKI I EGZISTENCIJA EKVILIBRIJUMA

U ovom radu dokazana je teorema selekcije Karateodorijevog tipa. Kao primena dobijena je teorema o stohastičkoj nepokretnoj tački. Rad sadrži i rezultat o postojanju ekvilibrija u abstraktnoj ekonomiji sa merljivim prostorom agenata i beskonačno dimenzionalnim prostorom strategije.

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COMMON FIXED POINTS FOR NONEXPANSIVE TYPE MAPPINGS IN CONVEX AND PROBABILISTIC CONVEX METRIC SPACES

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ABSTRACT

In this paper some common fixed point theorems in convex and probabilistic convex metric spaces are proved.

1. INTRODUCTION

Takahashi [1] introduced the notion of convexity in metric spaces and generalized some fixed point theorems in Banach spaces. Subsequently, Itoh [2], Tallman [3], Naimpally, Singh and Whitfield [4], Rhoades, Singh and Whitfield [5], and Hadžić [6] have studied convex metric spaces and fixed point theorems.

In this paper, we shall first introduce the concept of starshaped subsets in a convex and probabilistic convex metric space. Then we shall show some fixed point theorems for commuting mappings of the nonexpansive type on a starshaped subset of convex and probabilistic convex metric spaces. Our

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theorems improve and generalize the corresponding results in [6, 9, 10, 11].

2. DEFINITIONS AND LEMMAS

Definition 1. *Let* X *be a metric space and* $I = [0,1]$. *A mapping* $W : X \times X \times I \to X$ *is said to be a convex structure on* X *if for every* $(x,y,\lambda) \in X \times X \times I$ *and* $u \in X$

$$d(u,W(x,y,\lambda)) \leq \lambda d(u,x) + (1-\lambda)d(u,y). \tag{1}$$

X *together with a convex structure* W *is called a convex metric space.*

In the following, we assume throughout that X is a convex metric space.

Definition 2. *A nonempty subset* K *of* X *is said to be convex if for each* $(x,y,\lambda) \in X \times X \times I$, $W(x,y,\lambda) \in K$.

Definition 3. *A nonempty subset* K *of* X *is called starshaped if there exists a* $x_0 \in K$ *such that the set* $\{W(x,x_0,\lambda) : x \in K, \lambda \in I\} \subset K$. *The* x_0 *is said to be a star-centre of* K.

Clearly, every convex set is a starshaped set and the inverse is not true.

Definition 4. *Let* K *be a starshaped subset of* X *with a star-centre* x_0. *We say that* K *satisfies condition* (β) *if for all* $(x,y,\lambda) \in K \times K \times I$

$$d(W(x,x_0,\lambda),W(y,x_0,\lambda)) \leq \lambda d(x,y).$$

Definition 5. *A mapping* $S : X \to X$ *is said to be* (W,x_0)*-convex if for each* $(x,\lambda) \in X \times I$, $W(Sx,x_0,\lambda) = S(W(x,x_0,\lambda))$.

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Lemma 1. *Let* K *be a starshaped subset of* X *with a star-centre* x_0 *and let* S : K → K *be a* (W,x_0)*-convex mapping. Then the fixed point set fix* (S) *of* S *is a starshaped set with a star-centre* x_0.

Proof. By (1) we have $x_0 = W(x_0,x_0,\lambda)$, $\forall\lambda \in I$ $W(Sx_0,x_0,0) = x_0$. It follows from definition 5 that $Sx_0 = S(W(x_0,x_0,\lambda)) = W(Sx_0,x_0,\lambda)$ $\forall\lambda \in I$. Putting $\lambda = 0$, we obtain $x_0 = Sx_0$ and so $x_0 \in fix(S)$. Now for any $x \in fix(S)$ and $\lambda \in I$, we have

$$W(x,x_0,\lambda) = W(Sx,x_0,\lambda) = S(W(x,x_0,\lambda)).$$

Thus the set $\{W(x,x_0,\lambda) : x \in fix(S),\ \lambda \in I\} \subset fix(S)$, i.e. $fix(S)$ is a starshaped set with the star-centre x_0.

Definition 6. *A continuous mappings* f : K → K *is said to be* α*-densifying if for any bounded* D ⊂ K *with* α(D)>0,

$$\alpha(f(D)) < \alpha(D),$$

where α *is the Kuratowski measure of noncompactness.*

For terminologies, notations and properties of Menger spaces (X,F,t), the reader may consult [7].

Definition 7. *Let* (X,F,t) *be a Menger space. A mapping* W : X × X × I → X *is said to be a convex structure if for all* (x,y,u,λ) ∈ X × X × X × (0,1),

$$F_{u,W(x,y,\lambda)}(2\varepsilon) \geq t(F_{u,x}(\tfrac{\varepsilon}{\lambda}),F_{u,y}(\tfrac{\varepsilon}{1-\lambda})),$$

for each ε ≥ 0 *and* W(x,y,0) = y, W(x,y,1) = x.

Definition 8. *A nonempty set* K ⊂ X *is said to be a starshaped subset of a Menger space* (X,F,t) *with a convex struc-*

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ture W if there exists a $x_0 \in K$ *such that the set* $\{W(x,x_0,\lambda) : x \in K, \lambda \in I\} \subset K$. *The* x_0 *is called a star--centre of* K.

Definition 9. *A starshaped subset K of Menger space* (X,F,t) *with a convex structure* W *satisfies condition* (β) *if for all* $(x,y,\lambda) \in X \times X \times (0,1)$,

$$F_{W(x,x_0,\lambda),W(y,x_0,\lambda)}(\lambda\varepsilon) \geq F_{x,y}(\varepsilon).$$

for every $\varepsilon \geq 0$.

Definition 10. *Let* (X,F,t) *be a Menger space with a convex structure* W. *A mapping* $S : X \to X$ *is said to be* (W,x_0)*--convex if for each* $(x,\lambda) \in X \times I$, $W(Sx,x_0,\lambda) = S(W(x,x_0,\lambda))$.

Using a similar argument as in Lemma 1, we can easily prove

Lemma 2. *Let* (X,F,t) *be a Menger space with a convex structure* W, *and let* K *be a starshaped subset of* X *with a star-centre* x_0. *If* $S : K \to K$ *is a* (W,x_0)*-convex mapping, then the fixed point set* $fix(S)$ *of* S *is a starshaped subset with a star-centre* x_0.

3. MAIN RESULTS

Theorem 1. *Let* K *be a closed starshaped subset of a complete convex metric space* (X,d) *with a star-centre* x_0 *and* K *satisfy condition* (β). *Suppose that* $f : K \to K$ *is a non-expansive mapping, i.e. for each* $x,y \in K$

$$d(fx,fy) \leq d(x,y),$$

f(K) *is bounded and there exists* $m \in N$ *(the set of all positive integers) such that* f^m *is* α-densifying on $\{W(x,x_0,\lambda) : x \in \in f(K), \lambda \in I\}$. *Then* f *has a fixed point in* K.

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Proof. Let $\{k_n\}_{n\in N}$ be a sequence of real numbers from (0,1) such that $\lim_{n\to\infty} k_n = 1$ and for each $n \in N$, define

$$f_n x = W(fx, x_o, k_n), \quad \forall x \in K.$$

Since K satisfies condition (β), we have that for all $x,y \in K$

$$d(f_n x, f_n y) = d(W(fx,x_o,k_n), W(fy,x_o,k_n))$$

$$\leq k_n d(fx,fy) \leq k_n d(x,y).$$

By Banach's contraction theorem, for each $n \in N$, there exists $x_n \in K$ such that $x_n = f_n x_n$. Furthermore,

$$d(x_n, fx_n) = d(f_n x_n, fx_n) = d(W(fx_n,x_o,k_n), fx_n)$$

$$\leq k_n d(fx_n, fx_n) + (1-k_n) d(fx_n, x_o)$$

and since f(K) is bounded, it follows that

$$\lim_{n\to\infty} d(x_n, fx_n) = 0.$$

Since f is nonexpansive, we have

$$d(x_n, f^m x_n) \leq \sum_{k=0}^{m-1} d(f^k x_n, f^{k+1} x_n) \leq m d(x_n, fx_n)$$

and so

$$\lim_{n\to\infty} d(x_n, f^m x_n) = 0. \tag{2}$$

Let us prove that the set $\{W(fx,x_o,\lambda) : x \in K, \ \lambda \in (0,1)\}$ is bounded.

This follows from the inequality

$$d(u, W(fx,x_o,\lambda)) \leq \lambda d(u,fx) + (1-\lambda) d(u,x_o)$$

for all $(u,x) \in K \times K$ and since f(K) is bounded. From $x_n = f_n x_n$,

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$\forall n \in N$, we have that $\{x_n\}_{n\in N} \subset \{W(fx,x_o,\lambda) : x \in K, \lambda \in (0,1)\}$ and so the set $\{x_n\}_{n\in N}$ is bounded. Furthermore, for any $\varepsilon > 0$ we have from (2) that

$$\alpha(\{x_n\}_{n\in N}) \le \alpha(B(f^m(\{x_n\}_{n\in N}),\varepsilon)) \le \alpha(f^m(\{x_n\}_{n\in N}))+\varepsilon$$

where

$$B(A,\varepsilon) = \{x \in K : d(x,A) < \varepsilon\} \quad (A \subset K) \text{ (see [8])}$$

and so

$$\alpha(\{x_n\}_{n\in N}) \le \alpha(f^m(\{x_n\}_{n\in N})).$$

This implies that $\alpha(\{x_n\}_{n\in N}) = 0$ and there exists a convergent subsequence $\{x_{n_k}\}_{k\in N}$. Let $\lim_{k\to\infty} x_{n_k} = x^*$. Then from

$$d(x^*,fx^*) \le d(x^*,x_{n_k}) + d(x_{n_k},fx_{n_k}) + d(fx_{n_k},fx^*)$$

and (2), since f is continuous, it follows that $x^* = fx^*$.

Theorem 2. *Let K be a closed starshaped subset of a complete convex metric space (X,d) with a star-centre x_o and K satisfy condition (β). Suppose that $f,g,S,T : K \to K$ such that S and T commute with f (or g), f(K) (or g(K)) is bounded and the following conditions are satisfied*

(i) *There exists $m \in N$ such that f^m is α-condensing on $\{W(x,x_o,\lambda) : x \in f(K), \lambda \in I\}$ and for all $x,y \in K$, $d(fx,gy) \le (Sx,Ty)$,*

(ii) *S and T are (W,x_o)-convex and continuous.*

Then there exists $x^* \in K$ such that $x^* = fx^* = gx^* = Sx^* = Tx^*$.

Proof. Let $\mathcal{F}ix(S,T)$ denote the set of common fixed points of S and T. Since S and T are (W,x_o)-convex, it follows from Lemma 1 that $x_o \in \mathcal{F}ix(S,T)$ and $\mathcal{F}ix(S,T) \subset K$ is a starshaped subset of X with a star-centre x_o. By the continuity S and T, $\mathcal{F}ix(S,T)$ is also closed. From (i) we have that for all $x,y \in \mathcal{F}ix(S,T)$

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$$d(fx,gy) \leq d(x,y)$$

and hence $fx = gx$ for all $x \in \mathit{fix}(S,T)$ and for all $x,y \in \in \mathit{fix}(S,T)$

$$d(fx,fy) \leq d(x,y).$$

For each $x \in \mathit{fix}(S,T)$, since S and T commute with f, we have $fx = fSx = Sfx$ and $fx = fTx = Tfx$ and so $fx = gx \in \mathit{fix}(S,T)$. Hence, it follows from Theorem 1 that there exists x^* in $\mathit{fix}(S,T)$ such that $x^* = fx^*$ and so $x^* = fx^* = gx^* = Sx^* = Tx^*$.

Remark 1. It is easy to check that Theorem 1 of [6] is a very special case of Theorem 2.

Theorem 3. *Let* K *be a closed starshaped subset of a linear space* X *with a translation invariant metric satisfying* $d(\lambda x + (1-\lambda)y,0) \leq \lambda d(x,0) + (1-\lambda)d(y,0)$ *and* (X,d) *is complete. Suppose that* f, g, S, T : K → K *such that* S *and* T *commute with* f *(or* g*),* f(K) *(or* g(K)*) is bounded and the following conditions are satisfied*

(i) *There exists* $m \in N$ *such that* f^m *is* α*-densifying on* $\{W(x,x_o,\lambda) : x \in f(K), \lambda \in I\}$ *and for all* $x,y \in K$, $d(fx,gy) \leq d(Sx,Ty)$.

(ii) S *and* T *are continuous such that for each* $(x,\lambda) \in \in X \times I$
$S(\lambda x + (1-\lambda)x_o) = \lambda Sx + (1-\lambda)x_o$,
$T(\lambda x + (1-\lambda)x_o) = \lambda Tx + (1-\lambda)x_o$,
where x_o *is a star-centre of* K.

Then there exists $x^* \in K$ *such that* $x^* = fx^* = gx^* = Sx^* = Tx^*$.

Proof. We define the mapping $W : X \times X \times I \to X$ by

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$$W(x,y,\lambda) = \lambda x + (1-\lambda)y.$$

It is easy to check that W is a convex structure on X and so X is a complete convex metric space. Hence Theorem 3 follows from Theorem 2.

Remark 2. Theorem 3 improves and generalizes Theorem 2 of [9] and the corresponding results in [10, 11].

Theorem 4. *Let* K *be a closed starshaped subset of a complete Menger space* X *with a convex structure* W *which satisfies condition* (β). *Suppose that* $f : K \to K$ *is such that* $f(K)$ *is probabilistic bounded (which means that* $\sup_{\varepsilon \geq 0} D_{f(K)}(\varepsilon) = 1$ *where* $D_{f(K)}(\varepsilon) = \sup_{r<\varepsilon} \inf_{p,q \in f(K)} F_{p,q}(r)$, $\varepsilon \geq 0$), *for some* $m \in N$, f^m *is precompact on the set* $\{W(x,x_0,\lambda) : x \in f(K), \ \lambda \in (0,1)\}$ *where* x_0 *is a star-centre of* K *and for all* $x,y \in K$ *and* $\varepsilon \geq 0$

$$F_{fx,fy}(\varepsilon) \geq F_{x,y}(\varepsilon). \tag{3}$$

Then f *has a fixed point in* K.

Proof. *Let* $\{k_n\}_{n \in N}$ be a sequence satisfying the condition in Theorem 1, and for each $n \in N$, define $f_n x = W(fx, x_0, k_n)$ for all $x \in K$. Since K satisfies condition (β) from (3), we have that

$$F_{f_n x, f_n y}(k_n \varepsilon) = F_{W(fx,x_0,k_n),W(fy,x_0,k_n)}(k_n \varepsilon)$$

$$\geq F_{fx,fy}(\varepsilon) \geq F_{x,y}(\varepsilon),$$

for all $x,y \in K$ and $n \in N$. Since $f(K)$ is probabilistic bounded it is easy to check that the set $\{W(fx,x_0,k_n) : x \in K\}$ is also probabilistic bounded for each $n \in N$. From Theorem 11.2.3 of [12] it follows that for each $n \in N$, there exists $x_n \in K$ such that $x_n = f_n x_n$. Furthermore,

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$$F_{x_n,fx_n}(2\varepsilon) = F_{f_n x_n,fx_n}(2\varepsilon) = F_{W(fx_n,x_o,k_n),fx_n}(2\varepsilon)$$

$$\geq t(F_{fx_n,fx_n}(\tfrac{\varepsilon}{k_n}),F_{fx_n,x_o}(\tfrac{\varepsilon}{1-k_n})) = t(1,F_{fx_n,x_o}(\tfrac{\varepsilon}{1-k_n}))$$

$$= F_{fx_n,x_o}(\tfrac{\varepsilon}{1-k_n}).$$

Since f(K) is probabilistic bounded, for each $z \in K$ we have

$$\inf_{n\in N} F_{fx_n,fz}(\varepsilon) \geq \sup_{r<\varepsilon} \inf_{p,q\in f(K)} F_{p,q}(r) = D_{f(K)}(\varepsilon)$$

and so

$$(4) \qquad \sup_{\varepsilon\geq 0} \inf_{n\in N} F_{fx_n,fz}(\varepsilon) = \sup_{\varepsilon\geq 0} D_{f(K)}(\varepsilon) = 1$$

Since

$$(5) \qquad F_{fx_n,x_o}(\tfrac{\varepsilon}{1-k_n}) \geq t\left(F_{fx_n,fz}\left(\frac{\varepsilon}{2(1-k_n)}\right),F_{fz,x_o}\left(\frac{\varepsilon}{2(1-k_n)}\right)\right)$$

using the continuity of t, relations (4) and (5), and relation $\lim_{n\to\infty} k_n = 1$, we obtain that $\lim_{n\to\infty} F_{fx_n,x_o}(\frac{\varepsilon}{1-k_n}) = 1$ and so for each $\varepsilon > 0$,

$$(6) \qquad \lim_{n\to\infty} F_{x_n,fx_n}(\varepsilon) = 1.$$

Since for each $n \in N$, $x_n = f_n x_n = W(fx_n,x_o,k_n) \in \{W(x,x_o,\lambda) : x \in f(K), \lambda \in (0,1)\}$ and f^m is precompact on $\{W(x,x_o,\lambda) : x \in f(K), \lambda \in (0,1)\}$, there exists a subsequence $\{x_{n_k}\}_{k\in N}$ of $\{x_n\}_{n\in N}$ such that $\lim_{k\to\infty} f^m x_{n_k} = x^* \in K$. By (3), for each $\varepsilon \geq 0$, each $n \in N$ and each $k \in N$, we have that

$$F_{f^k x_n,f^{k+1}x_n}(\varepsilon) \geq F_{x_n,fx_n}(\varepsilon)$$

and so

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$$F_{x_n,f^m x_n}(\varepsilon) \geq t\Big(F_{x_n,fx_n}(\tfrac{\varepsilon}{2}), t(F_{x_n,fx_n}(\tfrac{\varepsilon}{2^2}), \ldots$$
$$\ldots, F_{x_n,fx_n}(\tfrac{\varepsilon}{2^{m-1}})\Big) \ldots\Big).$$

Hence, from $t(1,1) = 1$, the continuity of t and (6), we obtain that

$$\lim_{n\to\infty} F_{x_n,f^m x_n}(\varepsilon) = 1, \quad \forall \varepsilon > 0. \tag{7}$$

The continuity of t, relation (7) and the inequality

$$F_{x_{n_k},x^*}(\varepsilon) \geq t\Big(F_{x_{n_k},f^m x_{n_k}}(\tfrac{\varepsilon}{2}), F_{x^*,f^m x_{n_k}}(\tfrac{\varepsilon}{2})\Big)$$

imply that $\lim_{k\to\infty} x_{n_k} = x^*$. From the inequality

$$F_{x^*,fx^*}(\varepsilon) \geq t\Big(F_{x^*,x_{n_k}}(\tfrac{\varepsilon}{2}), t\Big(F_{x_{n_k},fx_{n_k}}(\tfrac{\varepsilon}{4}),$$
$$F_{fx_{n_k},fx^*}(\tfrac{\varepsilon}{4})\Big)\Big)$$
$$\geq t\Big(F_{x^*,x_{n_k}}(\tfrac{\varepsilon}{2}), t(F_{x_{n_k},fx_{n_k}}(\tfrac{\varepsilon}{4}), F_{x_{n_k},x^*}(\tfrac{\varepsilon}{4}))\Big),$$

it follows that $F_{x^*,fx^*}(\varepsilon)=1$ for each $\varepsilon > 0$ and so $x^* = fx^*$.

Theorem 5. *Let K be a closed starshaped subset of a complete Menger space X with a convex structure W which satisfies condition (β). Suppose that* $f,g,S,T : K \to K$ *are such that S and T commute with f (or g), f(K) (or g(K)) is probabilistic bounded and the following conditions are satisfied:*

(i) There exists $m \in N$ *such that* f^m *(or* g^m*) is precompact on the set* $\{W(x,x_0,\lambda) : x \in f(K), \lambda \in (0,1)\}$ *(or* $\{W(x,x_0,\lambda) : x \in g(K), \lambda \in (0,1)\}$*) and for all* $x,y \in K$

$$F_{fx,gy}(\varepsilon) \geq F_{Sx,Ty}(\varepsilon), \tag{8}$$

(ii) S *and* T *are continuous and* (W,x_0)*-convex*

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Then there exists $x^* \in K$ *such that* $x^* = fx^* = gx^* = Sx^* = Tx^*$.

Proof. Let $\mathcal{fix}(S,T)$ be the set of common fixed points of S and T. Since S and T are (W,x_0)-convex, from Lemma 2, it follows that $x_0 \in \mathcal{fix}(S,T)$ and $\mathcal{fix}(S,T)$ is a stars-haped subset of X with a star-centre x_0. By the continuity of of S and T, $\mathcal{fix}(S,T)$ is also closed. From (8) we have that for all $x,y \in \mathcal{fix}(S,T)$

$$F_{fx,gy}(\varepsilon) \geq F_{x,y}(\varepsilon), \text{ for all } \varepsilon \geq 0,$$

and hence for all $x \in \mathcal{fix}(S,T)$, $fx = gx$ and so all $x,y \in \in \mathcal{fix}(S,T)$

$$F_{fx,fy}(\varepsilon) \geq F_{x,y}(\varepsilon), \forall \varepsilon \geq 0.$$

Since S and T commute with f, for each $x \in \mathcal{fix}(S,T)$ we have that $fx = fSx = Sfx$ and $fx = fTx = Tfx$ and so $fx \in \mathcal{fix}(S,T)$. Thus, from Theorem 4 it follows that there exists $x^* \in \mathcal{fix}(S,T)$ such that $x^* = fx^*$ and so $x^* = fx^* = gx^* = Sx^* = Tx^*$.

Remark 3. Theorem 5 improves and generalizes Theorem 2 of [6].

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REZIME

ZAJEDNIČKE NEPOKRETNE TAČKE NEEKSPANZIVNIH PRESLIKAVANJA U KONVEKSNIM I VEROVATNOSNIM KONVEKSNIM METRIČKIM PROSTORIMA

U ovom radu dokazane su neke teoreme o zajedničkoj nepokretnoj tački u konveksnim i verovatnosnim konveksnim metričkim prostorima.

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SEQUENTIALLY CONDENSING MAPS

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ABSTRACT

A non-repulsive fixed point theorem is proved for upper semicontinuous, set-valued, acyclic and sequentially condensing transformations, T, of a convex, bounded, closed and infinite dimensional set, C, of a Banach space E into itself. It is also shown that in certain cases a sequentially condensing map can be made condensing with the choice of a suitable new measure of non-compactness.

1. INTRODUCTION

Let C be a convex, closed and bounded set of a Banach space E and let $T : C \to C$ be a single-valued or a set-valued map. It is known that if C is infinite dimensional and T is upper semicontinuous, condensing and such that T(x) is acyclic for every $x \in C$, then T has a non-repulsive fixed point (see [6], [3] and [1]).

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In the first part of this paper we show that the condition "T is condensing", i.e.

$$\alpha(T(A)) < \alpha(A) \tag{1.1}$$

for every $A \subset C$, $\alpha(A) \neq 0$, can be relaxed by assuming that (1.1) holds only for countable sets $A \subset C$, $\alpha(A) \neq 0$.

In the second part we obtain a result which shows that at least in certain cases, the above generalization is only apparent, since we can introduce a new measure of non-compactness, β, such that

$$\beta(T(A)) < \beta(A) \tag{1.2}$$

for every $A \subset C$, $\beta(A) \neq 0$.

Lemmas, Propositions and Theorems are stated and proved for set-valued maps, except in one case, (see Proposition 3.1), where the proof for single-valued maps is much simpler than the proof for set-valued maps (see Proposit. 3.2).

2. NOTATIONS AND DEFINITIONS

A set-valued map $T : C \to C$ is said to be *upper semicontinuous* if for every $x \in C$ the set $T(x)$ is compact; and for every neighborhood V of $T(x)$ there exists a neighborhood W of x such that

$$T(y) \subset V \tag{2.1}$$

for every $y \in W$.

We say that T is *acyclic* if for every $x \in C$ the set $T(x)$ is acyclic in the Čech cohomology with rational coefficients.

The map T is said to be *condensing* if for every $A \subset C$ with $\alpha(A) \neq 0$ we have

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(2.2) $$\alpha(T(A)) < \alpha(A)$$

where α is the Kuratowski measure of non-compactness [5], i.e.

(2.3) $\alpha(A) = \inf\{\varepsilon$: there is a finite covering of A with sets whose diameter does not exceed $\varepsilon\}$.

It is known that α has the following properties

(2.4)
- (i) $\alpha(A) = 0$ iff $\overline{A}$ is compact;
- (ii) $A \subset B \quad \alpha(A) \leq \alpha(B)$;
- (iii) $\alpha(A \cup B) = \max\{\alpha(A),\alpha(B)\}$;
- (iv) $\alpha(\overline{co}\ A) = \alpha(A)$ (see [2]), where $\overline{co}\ A$ is the closed convex hull of A;
- (v) $\alpha(tA) = |t|\alpha(A)$;
- (vi) $\alpha(A + B) \leq \alpha(A) + \alpha(B)$.

We shall say that $T : C \to C$ is *sequentially condensing* if (2.2) holds for countable sets $A \subset C$, $\alpha(A) \neq 0$. A condensing map T is sequentially condensing, but the converse may not be true. The advantage of replacing the previous condition "T is condensing" with the new one stems from the fact that the latter is more easily verifiable.

3. SEQUENTIALLY CONDENSING MAPS AND NON-REPULSIVE FIXED POINT THEOREMS

Lemma 3.1. *Let* $f : C \to C$ *be sequentially condensing and continuous. Then there exists a nonempty compact set* K *such that*

$$f(K) = K.$$

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Proof. Let $x_0 \in C$. Consider the sequence

$$x_0, x_1 = f(x_0), \ldots, x_n = f(x_{n-1}), \ldots .$$

Define $A = \bigcup_{n=0}^{\infty} f^n(x_0)$, $f^0(x_0) = x_0$. Then A is compact (2.4.i) and $f(A) \subset A$. Hence (see [7]) there exists $K \subset \bar{A}$ such that $f(K) = K$.

Proposition 3.1. *Let $f : C \to C$ be sequentially condensing and continuous. Define $C_\infty = \bigcap_{n=0}^{\infty} C_n$ where $C_0 = C$, $C_{n+1} = \overline{co}\, f(C_n)$.*
Then C_∞ is closed, convex and not empty.

Proof. Note that $K \subset C_\infty$.

We now take up the multivalued case. We again want to show that C_∞ is not empty.

Proposition 3.2. *Let $T : C \to C$ be a sequentially condensing and upper semicontinuous map. There C_∞ is closed, convex and not empty.*

Proof. The only non-trivial property of C_∞ is that $C_\infty \neq \emptyset$. To prove it consider the set $A = \bigcup_{n=0}^{\infty} T^n(x_0)$ where $x_0 \in C$. Since the image of a compact set under an upper semicontinuous map is compact we see that A is separable. Let $\{y_n\} \subset \bar{A}$ be a dense sequence in $\bar{A}$ and let $\{x_n\} \subset A$ be such that $d(x_n, y_n) \to 0$ as $n \to +\infty$. Then for every $\varepsilon > 0$ there exists $n(\varepsilon)$ such that

$$\bigcup_n x_n \subset B\Big(\bigcup_n y_n, \varepsilon\Big) \cup \Big(\bigcup_{i=1}^{n(\varepsilon)} x_i\Big).$$

It follows that $\alpha(\{x_n\}) = \alpha(\{y_n\})$. Obviously $x_n \in T^p(x_0)$ for some p, i.e., there exists $w_n \in T^{p-1}(x_0)$ such that $x_n \in T(w_n)$. Assume $\alpha(A) > 0$. Then

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$$\alpha(A) = \alpha(\overline{A}) = \alpha(\{y_n\}) = \alpha(\{x_n\}) \leq \alpha(\bigcup T(w_n)) <$$

$$< \alpha(\{w_n\}) \leq \alpha(A).$$

This contradiction shows that $\overline{A}$ is compact. This implies that if we select a sequence $z_0 = x_0$, $z_1 \in T(x_0)$, $z_2 \in T^2(x_0)$,... ; then there exists $n_p \to +\infty$ such that $z_{n_p} \to x$ for some $\overline{x} \in X$. From $z_{n_p} \in C_{n_p}$ and from $C_n \supset C_{n+1}$ we obtain $\overline{x} \in C_n$ for every n. Hence $C_\infty \neq \emptyset$.

Theorem 3.1. *Let* $T : C \to C$ *be sequentially condensing and upper semicontinuous. Then* C_∞ *is invariant and compact.*

Proof. Obviously $T(C_0) \subset C_0$. Moreover, from $T(C_n) \subset C_n$ we can easily get $T(C_{n+1}) \subset C_{n+1}$. Hence C_∞ is invariant We also know that C_∞ is not empty. It remains to show that $\alpha(C_\infty) = 0$. We shall prove that for every sequence $\{x_n\} \subset C_\infty$ we have $\alpha(\{x_n\}) = 0$.

Let $Z = \{\{x_n\} : \exists\ n_p \to +\infty$ such that $x_n \in C_{n_p}\}$. Define

$$a = \sup\{\alpha(\{x_n\}) : \{x_n\} \in Z\} \tag{3.1}$$

and let $\{y_n\}_q \in Z$ be a sequence of elements in Z such that

$$\alpha(\{y_n\}_q) \to a \text{ as } q \to +\infty.$$

Let

$$M = \bigcup_q (\bigcup_n y_{nq}) \tag{3.2}$$

where y_{nq} is the n-th element of the q-th sequence. M is countable and obviously can be rearranged in a sequence $\{w_m\} \in Z$. Moreover, $\alpha(\{w_m\}) = a$.

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Assume $a > 0$ and let $m_r \to +\infty$ be such that $w_m \in \overline{co}\, T(C_{m_r-1})$ for every m. Let $z_m \in co\, T(C_{m_2-1})$ be such that

$$\|w_m - z_m\| = \|y_m\| \leq \frac{1}{m}.$$

Then $\{y_n\}$ is convergent and $\alpha(\{z_m\}) = \alpha(\{w_m\})$. Since $z_m \in co\, T(C_{m_r-1})$ there exist finitely many points $x_{m_1},\dots,x_{m_{j_m}}$ in C_{m_r-1} such that

$$z_m \in co\Big(\bigcup_{i=0}^{j_m} T(x_{m_i})\Big).$$

Let

$$N = U\Big(\bigcup_{i=0}^{j_m} x_{m_i}\Big). \tag{3.3}$$

Then N is countable and it can be considered as an element of Z. Hence

$$a = \alpha(\{z_m\}) \leq \alpha(co(T(N))) = \alpha(T(N)) < \alpha(N) = a.$$

This contradiction shows that $a = 0$.

The basic ideas of the previous proof are derived from P. Massat [8].

We are now ready to prove the main result of this first part. Recall that a point $x_o \in C$ such that $x_o \in T(x_o)$ is said to be *repulsive* if there exists a neighborhood V of x_o such that for every neighborhood W of x_o there exists n such that

$$T^n(V \setminus W) \cap V = \emptyset. \tag{3.4}$$

This notion was first introduced by F. Browder [1].

Theorem 3.2. *Let $T : C \to C$ be an upper semicontinuous set-valued and sequentially condensing map. Assume that T*

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is acyclic and dim C = ∞. Then T has a non-repulsive fixed point.

Proof. Let $P \subset C$ be an infinite dimensional and compact set. Define

$$C_0 = C \cup P = C$$
$$C_n = \overline{co}(T(C_{n-1}) \cup P)$$
$$C_\infty = \bigcap_n C_n \quad .$$

Then C_∞ is non empty, closed, convex, infinite dimensional and invariant. Moreover, with a minor modification in the proof of Theorem 3.1 it can be shown that C_∞ is compact. Hence (see [3]) T has a non-repulsive fixed point in C_∞, which is obviously non-repulsive in C.

Remark 3.2. In the case of single valued, continuous maps Theorem 3.2 contains Sadovskii's [10] result as a particular case. In [10] the map f is assumed to be condensing and the principle of transfinite induction is used. Subsequently several authors (see [4], [7], [8], [9]) succeded in giving proofs of Sadovskii result without using the principle of transfinite induction. The first proofs of this type are due to M. Martelli [7] and M. Furi-A. Vignoli [4]. The following construction provides another proof which is noteworthy for its simplicity.

Let $x_0 \in C$. Define $A_0 = x_0$, $A_1 = \overline{co}(x_0 \cup f(x_0))$, $A_2 = \overline{co}(A_1 \cup f(A_1)), \ldots, A_n = \overline{co}(A_{n-1} \cup f(A_{n-1})), \ldots$. We see that

$$A_1 \subset A_2 \subset \ldots \subset A_n \subset \ldots$$

Therefore the set $A = \bigcup_n A_n$ is convex. Moreover $f(A) \subset A$, since $f(A_i) \subset A_{i+1}$. By the continuity of f we obtain $f(\overline{A}) \subset \overline{A}$. It remains to show that A is compact. This property of $\overline{A}$ is established by observing that

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$$A \subset \overline{co}(x_0 \cup f(A)).$$

Notice that the above construction cannot be extended to upper semicontinuous multivalued maps, T, since $T(A) \subset A$ does not imply $T(\overline{A}) \subset \overline{A}$.

Remark 3.3. Properties (iii), (iv), and (vi) of the α-measure of noncompactness are never used in the Lemmas, Propositions and Theorems of this part.

4. β-MEASURE OF NON-COMPACTNESS

Let $T : C \to C$ be an upper semicontinuous set-valued map. Define the sequence of sets $\{C_n\}$ as in 3, and assume that $\alpha(C_n) \to 0$ as $n \to +\infty$. Under this assumption we want to show that we can define a new measure of non-compactness, β, such that $\beta(T(A)) < \beta(A)$ for every $A \subset C$, $\beta(A) \neq 0$.

We start by selecting a sequence $\{a_n\}$ of real numbers such that

$$a_n < a_{n+1}, \quad a_0 = 1, \quad \lim_{n\to+\infty} a_n = 2.$$

Given $M \subset C$ we consider the sequence of sets

$$M_0 = \overline{co}\, M, \quad M_1 = \overline{co}\, T(M_0), \ldots$$

and we define

$$\beta(M) = \max\{a_n \alpha(M_n) : n = 0,1,\ldots\}.$$

Let us now study the properties of β.

Proposition 4.1. $\beta(M) = 0$ *iff* $\overline{M}$ *is compact and* $\beta(co\ M) = \beta(M)$.

Proof. If $\beta(M) = 0$ then $\alpha(M) = 0$ and $\overline{M}$ is compact.

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If $\overline{M}$ is compact then $T(M_0)$ is compact since T is upper semi-continuous. Therefore M_1 is compact, and, in general, M_n is compact for every n. Thus $\beta(M) = 0$.

The second property is evident.

It is also obvious that $M \subset N$ implies $\beta(M) < \beta(N)$ and if T is positively homogeneous then $\beta(tM)=t\beta(M)$ for $t>0$.

Theorem 4.1. *$T : C \to C$ is β-condensing, i.e. $\beta(T(M)) < \beta(M)$ for every non-compact set $\overline{M} \subset C$.*

Proof. Define $N = T(M)$. Then $N_0 = \overline{co}\, T(M) \subset \overline{co}\, T(M_0) = M_1$, and in general,

$$N_i \subset M_{i+1}$$

Since the sequence $\{a_n\}$ is increasing we have $\beta(M) = max\{a_n\alpha(M_n): n = 0,1,...\} > max\{a_{n-1}\alpha(M_n) : n = 1,2,...\} \geq max\{a_n\alpha(N_n) : n = 0,1,...\} = \beta(T(M))$.

Remark 4.1. We have presented our results in the context of Banach spaces, but it can be easily shown that they hold in complete, locally convex topological linear spaces, at least to the extent of proving that f has a fixed point. The definition of α needs to be suitably modified and the Tichonov fixed point theorem will be used once a compact, convex and invariant set C has been produced.

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REZIME

SEKVENCIJALNO KONDENZUJUĆA PRESLIKAVANJA

Nerepulzivna teorema o nepokretnoj tački je dokazana za od gore neprekidna, višeznačna, aciklična i sekvencijalno kondenzujuća preslikavanja T, konveksnog, ograničenog, zatvorenog i beskonačno dimenzionalnog skupa C u samog sebe, gde je $C \subset E$ a E je Banahov prostor. Pokazano je da u odredjenim slučajevima sekvencijalno kondenzujuća preslikavanja postaju kondenzujuća u odnosu na novu meru nekompaktnosti.

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THE FRENET FORMULAE OF THE RIEMANN-OTSUKI SPACE

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ABSTRACT

In this paper the Frenet formulae of the Riemann-Otsuki space with respect to covariant and contravariant part of the connection are obtained.

INTRODUCTION

The basis of the theory of Otsuki spaces has been laid down by T. Otsuki and A. Moór. The metric used determined that the observed space is of Weyl-Otsuki's or of Riemann-Otsuki's kind. In this paper we shall consider the Riemann-Otsuki space and we shall determine the Frenet formulae with respect to the co-resp. contravariant part of the connection. According to the following observation, we get that only in the contravariant part of the connection the Frenet formulae of the $R\text{-}O_n$ space are different from the known Frenet formulae of Riemannian geometry. The difference came from the fact that in Otsuki spaces $D\delta^i_j \neq 0$ holds.

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In all Otsuki spaces we have, with respect to the local coordinates x^i of an n-dimensional differentiable manifold, an *a-priori* given tensor P^i_j such that $det\|P^i_j\| \neq 0$ holds and the inverse tensor Q^i_j exists so that $P^i_j Q^j_r = \delta^i_r$. In the metric Otsuki spaces the metric tensor g_{ij} $(det\|g_{ij}\| \neq 0)$ is given so that in the W - O_n Weyl-Otsuki space $\nabla_k g_{ij} = \gamma_k g_{ij}$, but in the R-$O_n$ (Riemann-Otsuki space)

$$\nabla_k g_{ij} = 0 \tag{0.1}$$

holds. In Otsuki spaces the covariant differential of the tensor T^i_j is defined by

$$DT^i_j = P^i_a P^b_j \overline{D} T^a_b = P^i_a P^b_j (\partial_k T^a_b + {}'\Gamma_r{}^a{}_k T^r_b - {}''\Gamma_b{}^r{}_k T^a_r) dx^k. \tag{0.2}$$

The Leibnitz formula *does not hold* for this differential. The differential $\overline{D}$ is the *basic* covariant differential. The different coefficients of the connection are characteristic of the Otsuki spaces, and here are

$$\delta^i_j|_k = {}'\Gamma_j{}^i{}_k - {}''\Gamma_j{}^i{}_k \neq 0. \tag{0.3}$$

The coefficient of the connection ${}''\Gamma_j{}^i{}_k$ was determined from the relation (0.1) and the coefficients of connection ${}'\Gamma_j{}^i{}_k$ are got from

$$\partial_k P^i_j + {}''\Gamma_a{}^i{}_k P^a_j - P^i_a {}'\Gamma_j{}^a{}_k = 0. \tag{0.4}$$

This relation is known as *Otsuki's relation*.

In Otsuki spaces it is possible to determine the covariant differentials D and $\overline{D}$ with respect only to the co- -resp. contravariant part of the connection. So

$$\overline{D}T^i_j := {}'\nabla_k T^i_j dx^k = (\partial_k T^i_j + {}'\Gamma_r{}^i{}_k T^r_j - {}'\Gamma_j{}^r{}_k T^i_r) dx^k \tag{0.5}$$

holds. For this basic covariant differential the Leibnitz

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formula holds. The basic covariant differential $''\overline{D}$ can be defined in the same way. It is characteristic that the basic covariant differential $'\overline{D}$ is identical in the case of contravariant indices with the basic covariant differential $\overline{D}$, and similarly in the case of covariant indices the basic covariant differential $''\overline{D}$ is identical with the basic covariant differential $\overline{D}$.

In the following we shall use the relations

$$(0.6)\qquad '\overline{D}g_{ij} = dg_{ij} - ('\Gamma_i{}^r{}_k g_{rj} + '\Gamma_j{}^r{}_k g_{ir})dx^k,$$

$$(0.7)\qquad ''\overline{D}g_{ij} = dg_{ij} - (''\Gamma_i{}^r{}_k g_{rj} + ''\Gamma_j{}^r{}_k g_{ir})dx^k = 0,$$

$$(0.8)\qquad '\overline{D}g^{ra} = - g^{ia}g^{jr}('\overline{D}g_{ij}),$$

$$(0.9)\qquad ''\overline{D}g^{ra} = 0.$$

1. THE FRENET FORMULAE WITH RESPECT TO THE CONTRAVARIANT COMPONENTS OF THE VECTORS

Let the point P of the curve $C : x^i = x^i(s)$ be given where s is the arch length parameter. In that point $y_o^i := \frac{dx^i}{ds}$ are the components of the unit tangent vector $\underset{o}{y}$. Applying the basic covariant differential $'\overline{D}$ on the relation

$$(1.1)\qquad g_{ij}y_o^i y_o^j = 1$$

using the Leibnitz formula and the symmetry of the tensor g_{ij} we get

$$(1.2)\qquad ('\overline{D}g_{ij})y_o^i y_o^j + 2g_{ij}y_o^i('\overline{D}v_o^j) = 0.$$

From relations (0.6) and (0.7), it follows that $'\overline{D}g_{ij} = -2g_{r(i}\overline{D}\delta^r_{j)}$. Substituting it in relation (1.2) and using that $'\overline{D}v_o^j = \overline{D}v_o^j$, we get

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$$g_{ij}v_o^i(\bar{D}v_o^j - v_o^r\bar{D}\delta_r^j) = 0.$$

Let v_1^i be the components of the unit vector v_1 so that

(1.3) a) $v_1^j := \frac{1}{\kappa_1(s)}(\bar{D}v_o^j - v_o^r\bar{D}\delta_r^j), \kappa_1 > 0$; b) $g_{ij}v_1^iv_1^j = 1$

holds. From the above relations we can see, that $v_1 \perp v_o$ and

(1.4) $$\kappa_1(s) = \left(g_{rq}(\bar{D}v_o^r - v_o^a\bar{D}\delta_a^r)(\bar{D}v_o^q - v_o^b\bar{D}\delta_b^q)\right)^{\frac{1}{2}}$$

and

(1.5) $$\bar{D}v_o^j = \kappa v_{11}^j + v_o^q\bar{D}\delta_q^j$$

hold. (1.5) is the *first Frenet formula* of the basic covariant differential applied on the *contravariant* components of the vectors.

Applying now the basic covariant differential $\bar{D}$ on relation (1.3.b) with a calculation like the one above we get that $g_{ij}v_1^i(\bar{D}v_1^j - v_1^r\bar{D}\delta_r^j) = 0$. This means that the vector v_1 is orthogonal to the direction of $\bar{D}v_1^j - v_1^r\bar{D}\delta_r^j$, and if v_2^i denotes the components of the unit vector orthogonal on the plain determined by the vectors v_o and v_1, it follows that

(1.6) $$\bar{D}v_1^j - v_1^r\bar{D}\delta_r^j = \alpha v_o^j + \kappa v_{22}^j$$

holds. Now we shall use that $v_o \perp v_2$ and apply the covariant differential $\bar{D}$ on the relation $g_{ij}v_o^iv_1^j = 0$. A calculation like the one above with the substitution $\bar{D}v_o^j$ from (1.5) and $\bar{D}v_1^j$ from (1.6) according to (1.1) and (1.3 b) gives

(1.7) $$\alpha = -\kappa_1(s).$$

Substituting it in (1.6), we get

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(1.8) $$\overline{D}v_1^j = -\kappa_{10}v^j + \kappa_{22}v^j + v_1^r\overline{D}\delta_r^j.$$

From this relation there is $v_2^i = \frac{1}{\kappa_2(s)}(\overline{D}v_1^i + \kappa_{10}v^i - v_1^r\overline{D}\delta_r^i)$ and

$$\kappa_2(s) = \left(g_{ij}(\overline{D}v_1^j + \kappa_{10}v^j - v_1^r\overline{D}\delta_r^j)(\overline{D}v_1^i + \kappa_{10}v^i - v_1^b\overline{D}\delta_b^i)\right)^{\frac{1}{2}}.$$

Now we can formulate

Lemma 1. *If v is the unit vector, then* $(\overline{D}v^j - v^r\overline{D}\delta_r^j)g_{ij}v^i = 0$ *and* $(\overline{D}v_j - v_r\overline{D}\delta_j^r)g^{ij}v_i = 0$ *holds.*

(The proof of the covariant case follows in §2.)

We shall now make the following generalization.

Let for mutually orthogonal unit vectors v_ℓ ($\ell = 0,\ldots,p-1$) be

(1.9) $$\overline{D}v_\ell^j = -\kappa_\ell v_{\ell-1}^j + \kappa_{\ell+1}v_{\ell+1}^j + v_\ell^r\overline{D}\delta_r^j$$

so that $\kappa_0 = 0$, and if $q = 1,\ldots,p-1$ then

(1.10) $$\kappa_q = \left(g_{ij}(\overline{D}_{q-1}v^j + \kappa_{q-1}v_{q-2}^j - v_{q-1}^r\overline{D}\delta_r^j)(\overline{D}_{q-1}v^i + \kappa_{q-1}v_{q-2}^i - v_{q-1}^t\overline{D}\delta_t^i)\right)^{\frac{1}{2}}$$

holds. We construct the vector v_p^i ($p < n$) so that

(1.11) $$v_p^i := \frac{1}{\kappa_p}(\overline{D}_p v_{p-1}^i + \kappa_{p-2}v_{p-2}^i - v_{p-1}^r\overline{D}\delta_r^i).$$

According to the above Lemma $g_{ij}v_p^i(\overline{D}v_p^i - v_p^r\overline{D}\delta_r^i) = 0$ holds and we can to write the linear combination

(1.12) $$\overline{D}v_p^j - v_p^r\overline{D}\delta_r^j = \alpha_0 v_0^j + \alpha_1 v_1^j + \ldots + \alpha_{p-1}v_{p-1}^j + \kappa_{p+1}v_{p+1}^j.$$

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Contracting this with $g_{ij}\,{}^{v}_{\ell}{}^{i}$, using that ${}^{v}_{\ell} \perp {}^{v}_{m}$ $(m \neq \ell)$ and ${}^{v}_{\ell}$ is the unit vector, we get

$$\alpha_\ell = g_{ij}\,{}^{v}_{\ell}{}^{i}(\bar{D}\,{}^{v}_{p}{}^{j} - {}^{v}_{p}{}^{r}\bar{D}\delta^{j}_{r}).$$

With respect to (1.9) $\alpha_\ell = -g_{ij}{}^{v}_{\ell+1}{}^{i}\,{}^{v}_{p}{}^{j}\,{}^{\kappa}_{\ell+1}$ i.e. if $\ell \neq p-1$, then $\alpha_\ell = 0$, and if $\ell = p-1$, then $\alpha = -{}^{\kappa}_{p}$ holds. Substituting this in (1.12), it follows

$$\bar{D}\,{}^{v}_{p}{}^{j} = -{}^{\kappa}_{p}\,{}^{v}_{p-1}{}^{j} + {}^{\kappa}_{p+1}\,{}^{v}_{p+1}{}^{j} + {}^{v}_{p}{}^{2}\bar{D}\delta^{j}_{q} \tag{1.13}$$

and we can formulate

Theorem 1. *If* $C : x^i(s)$ *is the curve of an* $R - 0_n$ *space and* ${}^{v}_{\ell}$, $\ell = 0,\ldots,p-1$, $(p < n)$ *are mutually orthogonal unit vectors which satisfy the relation (1.9) and* ${}^{v}_{p+1}$ *is the unit vector orthogonal to all before and* ${}^{\kappa}_{o} = 0$, ${}^{\kappa}_{n} = 0$ *holds, then the vector* ${}^{v}_{p}$ *satisfies the relation (1.9), too.*

If we use Otsuki's covariant differential D, then from the connection $Dv^j = P^j_a \bar{D}v^a$ it follows that $\bar{D}v^a = Q^a_i Dv^i$. Applying this on (1.9), we get

$$D\,{}^{v}_{\ell}{}^{j} = P^j_i(-{}^{\kappa}_{\ell}\,{}^{v}_{\ell-1}{}^{i} + {}^{\kappa}_{\ell+1}\,{}^{v}_{\ell+1}{}^{i}) + {}^{v}_{\ell}{}^{q}Q^b_q D\delta^j_b \tag{1.14}$$

with respect to $\ell = 0,\ldots,(n-1)$; ${}^{\kappa}_{o} = 0$; ${}^{\kappa}_{n} = 0$. We can now state.

Theorem 2. *If in the point* M *of the curve* C *in the* $R - 0_n$ *space the mutually orthogonal unit vectors* ${}^{v}_{o}, {}^{v}_{1}, \ldots, {}^{v}_{n-1}$ *satisfying relations (1.9) and (1.10) so that* ${}^{\kappa}_{o} = 0$ *and* ${}^{\kappa}_{n} = 0$ *hold, then (1.14) is the Frenet formula of the curve* C *of the* $R - 0_n$ *space. This formula is applied with respect to the covariant differential* D *on the contravariant components of the observed vectors.*

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Remark 1. *The relation (1.14) is the Frenet formula with respect to the covariant differential* $'D$, *too, applied on the contravariant components of the vectors.*

One can see that the difference between (1.14) and the known formula of Riemannian geometry is the covariant differential of the Kronecker-δ. In such a case but not only such special ones where this differential is zero, the formula (1.14) reduces to the known Frenet formula multiplied with the tensor P^i_j.

Now we shall apply the basic covariant differential $''\overline{D}$ on (1.1). Using the Leibnitz formula, the symmetry of the tensor g_{ij} and (0.7), we get that $g_{ij}v^i_o(''\overline{D}v^j_o) = 0$, i.e. $v_o \perp ''\overline{D}v_o$ holds. This means that we can construct the unit vector $\overline{v}_1 \perp v_o$ so that

$$''\overline{D}v^j_o = \kappa^*_1\overline{v}^j_1, \tag{1.15}$$

where the scalar $\kappa^*_1(s)$ satisfies

$$\kappa^*_1(s) = (g_{ij}\,''\overline{D}v^i_o\,''\overline{D}v^j_o)^{\frac{1}{2}} > 0. \tag{1.16}$$

The relation (1.15) is *the first Frenet formula* of the basic covariant differential $''\overline{D}$ applied on the contravariant components of the vectors. Now we shall prove

Theorem 3. *From the connection between the basic covariant differentials* $'\overline{D}$ *and* $''\overline{D}$ *it follows that* $v_1 = \overline{v}_1$ *and the value of* κ_1 *is equivalent to the value of* κ^*_1.

Proof. It is easy to see that

$$''\overline{D}v^i_o = '\overline{D}v^i_o - v^q_o\overline{D}\delta^i_q \tag{1.17}$$

holds. Substituting this in (1.16) we get that according to (1.5) $\kappa_1(s) = \kappa^*_1(s)$ holds. In the following * by $\kappa_i(s)$ we shall

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denote that the curvature will be expressed with the aid of the basic covariant differential $''\overline{D}$.

According to the characteristics of the covariant differential $''D$, we can state

Theorem 4. *With respect to the basic covariant differential* $''\overline{D}$ *the Frenet formula of the curve* C *of the* $R - 0_n$ *space is not different from the known formula of the Riemannian space. If* $\underset{o}{v}, \underset{1}{v}, \ldots, \underset{n-1}{v}$ *are in point* P *of curve* C *in a suitable way constructed mutually orthogonal unit vectors, then*

$$(1.18) \qquad ''D\underset{\ell}{v}^j = P^j_i\left(-\kappa^*_{\ell}\ \underset{\ell-1}{v}^i + \kappa^*_{\ell+1}\ \underset{\ell+1}{v}^i\right)$$

is the Frenet formula with respect to the covariant differential applied on the contravariant components of the observed vectors.

2. THE FRENET FORMULAE WITH RESPECT TO THE COVARIANT COMPONENTS OF THE VECTORS

According to the definition $\underset{o}{v}_i = g_{ij}\frac{dx^j}{ds}$ holds and $\underset{o}{v}_i$ are the covariant components of the unit tangent vector $\underset{o}{v}$. Applying the basic covariant differential $''\overline{D}$ on the relation $g^{ij}\underset{o}{v}_i\underset{o}{v}_j = 1$, using the Leibnitz formula and relations (0.8),(0.6) and (0.7), we get $g^{ij}\underset{o}{v}_i(''\overline{D}\underset{o}{v}_j + \underset{o}{v}_b\overline{D}\delta^b_j) = 0$. This proved the second part of Lemma 1 from the first paragraph. Now again, as in the first paragraph, we can construct the mutually orthogonal unit vectors $\underset{\ell}{v}$ $(\ell = 0,1,\ldots,n-1)$, so that

$$(2.1) \qquad ''\overline{D}\,\underset{\ell}{\overline{v}}_j = -\kappa^{**}_{\ell}\,\underset{\ell-1}{\overline{v}}_j + {}_{\ell}\kappa^{**}_{+1}\,\underset{\ell+1}{\overline{v}}_j - \underset{\ell}{\overline{v}}_r\,\overline{D}\delta^r_j$$

holds with the remarks $\kappa^{**}_{o} = 0$, $\kappa^{**}_{n} = 0$, and if $\ell = 0,\ldots,n-2$ then

$$(2.2) \qquad {}_{\ell}\kappa^{**}_{+1} = \left(g^{ij}(''\overline{D}\underset{\ell}{\overline{v}}_j + \kappa^{**}_{\ell}\underset{\ell-1}{\overline{v}}_j + \underset{\ell}{\overline{v}}_r\overline{D}\delta^r_j)(''\overline{D}\underset{\ell}{\overline{v}}_i + \kappa^{**}_{\ell}\underset{\ell-1}{\overline{v}}_i + \underset{\ell}{\overline{v}}_q\overline{D}\delta^q_i)\right)^{\frac{1}{2}}.$$

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We can now formulate

Theorem 5. *From the relation* $\overline{v}_{\ell i} = g_{ij}\overline{v}^{j}_{\ell}$ *it follows that the value of* κ^{**}_{ℓ} *is equal to the value of* κ_{ℓ} *and* $\overline{v}_{\ell} \equiv v_{\ell}$ *holds.*

Proof. If $\ell = 0$ then according to $\overline{v}_{oi} \equiv v_{oi}$ and $v_{oi} = g_{ij}v^{j}_{o}$ using the Leibnitz formula from (2.2), we get

$$\kappa^{**}_{1} = \Big(g^{ij}((\overline{'D}g_{ja})v^{a}_{o} + g_{ja}(\overline{'D}v^{a}_{o}) + g_{ra}v^{a}_{o}\overline{D}\delta^{r}_{j})$$

$$((\overline{'D}g_{ib})v^{b}_{o} + g_{ib}(\overline{'D}v^{b}_{o}) + g_{sb}v^{b}_{o}\overline{D}\delta^{s}_{i})\Big)^{\frac{1}{2}}.$$

Since $\overline{'D}g_{ja} = -2g_{s(j}\overline{D}\delta^{s}_{a)}$ and (1.5) holds according to (1.3 b), it follows that $\kappa^{**}_{1} = \kappa_{1}$. Now we shall suppose that $\kappa^{**}_{\ell} = \kappa_{\ell}$ holds, and using the calculation as above, we get that

$$\kappa^{**}_{\ell+1} = (g_{ij}\overline{v}^{i}_{\ell+1}\overline{v}^{j}_{\ell+1}\kappa^{2}_{\ell+1})^{\frac{1}{2}}$$

Since $\overline{v}^{i}_{\ell+1}$ are the unit vectors here, the statement of the first part of the theorem follows.

To prove the second part we shall use that $v_{o} \equiv \overline{v}_{o}$. Now we shall suppose that $\overline{v}_{p} \equiv v_{p}$ $(p = 0,\ldots,\ell)$, and from (2.1), using the first part of the theorem, it follows that

$$\overline{v}_{\ell+1 i} = \frac{1}{\kappa_{\ell+1}}(\overline{'D}v_{\ell j} + \kappa_{\ell}v_{\ell-1 j} + v_{\ell r}\overline{D}\delta^{r}_{j}).$$

Since $v_{i} = g_{ij}v^{j}$, using the Leibnitz formula and the calculation as above, contracting by g_{ia}, we get that

$$\overline{v}^{r}_{\ell+1} = \frac{1}{\kappa_{\ell+1}}(\overline{'D}v^{r}_{\ell} + \kappa_{\ell}v^{r}_{\ell-1} - v^{a}_{\ell}\overline{D}\delta^{r}_{a}),$$

i.e. according to (1.11) $\overline{v}^{r}_{\ell+1} = v^{r}_{\ell+1}$ holds.

In the following κ^{**}_{ℓ} by κ_{ℓ} we shall denote that the curvature will be expressed with the aid of the basic covariant differential $\overline{'D}$ applied on the covariant components of

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the vectors as in (2.2).

Theorem 6. *If in the point* M *of the curve* C *in the* R-0_n *space the mutually orthogonal unit vectors* $\underset{o}{v},\underset{1}{v},\ldots \underset{n-1}{v}$ *are constructed so that* $\underset{p}{v}$ $(p = 1,\ldots,n-1)$ *satisfies*

$$\underset{pi}{v}: = \frac{1}{\underset{p}{\kappa}^{**}}(\overset{\backprime}{\underset{p-1}{D}}\underset{i}{v} + \underset{p-1}{\kappa}^{**}\underset{p-2}{\underset{i}{v}} + \underset{p-1}{\underset{r}{v}}\overline{D}\delta^r_i) \tag{2.3}$$

with

$$\underset{o}{\kappa}^{**} = 0, \quad \underset{n}{\kappa}^{**} = 0,$$

then

$$\overset{\backprime}{D}\underset{pi}{v} = P^j_i(-\underset{p}{\kappa}^{**}\underset{p-1}{\underset{i}{v}} + \underset{p+1}{\kappa}^{**}\underset{p+1}{\underset{i}{v}}) - \underset{pr}{v}D\delta^a_i Q^r_a \tag{2.4}$$

is the Frenet formula with respect to the covariant differential $\overset{\backprime}{D}$ *applied on the covariant components of the observed vectors.*

If we make the above calculation with respect to the basic covariant differential $\overset{\backprime\backprime}{D}$, according to relation (0.9) and the fact that in the case of covariant indices the basic covariant differentials $\overline{D}$ and $\overset{\backprime\backprime}{D}$ are not different, it follows that this case is not different from the observation of Riemannian space. We can only say

Remark 2. *The relation*

$$D\underset{pi}{v} = P^j_i(-\underset{p}{\kappa}^{***}\underset{p-1}{\underset{i}{v}} + \underset{p+1}{\kappa}^{***}\underset{p+1}{\underset{i}{v}}) \tag{2.5}$$

is the Frenet formula with respect to Otsuki's covariant differential D *applied on the covariant components of the vectors.*

Here ***, by scalars, denotes that the curvature will be expressed with the aid of Otsuki's basic covariant differential $\overline{D}$ applied on the contravariant components of the vectors.

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From (2.5). it follows that

$$(2.6)\qquad {}^{\kappa***}_{p+1} = \left(g^{ij}({}^{\overline{D}v_i}_{p^i} + {}^{\kappa***}_{p}\, v_{i\,p-1})({}^{\overline{D}v_j}_{p^j} + {}^{\kappa***}_{p}\, v_{j\,p-1})\right)^{\frac{1}{2}}$$

holds. Using connection $\overline{D}v_i = \acute{\overline{D}}v_i + v_r \overline{D}\delta^r_i$, we get

$$ {}^{\kappa***}_{p+1} = \left(g^{ij}({}^{\acute{\overline{D}}v_i}_{p^i} + {}^{\kappa***}_{p}\, v_{i\,p-1} + v_{p^r}\overline{D}\delta^r_i)({}^{\acute{\overline{D}}v_j}_{pj} + \right.$$

$$\left. + {}^{\kappa***}_{p}\, v_{j\,p-1} + v_{p^q}\overline{D}\delta^q_j)\right)^{\frac{1}{2}}.$$

This means that the value of the curvatures is not different in the case of different differentials. This result can be expected because the curvature depends only on the curve and on a suitably constructed vector frame which is unequally determined. The Frenet formulae are different from the known formulae of Riemannian space only if $D\delta^i_j \neq 0$ holds, i.e. if the basic covariant differential of the metric tensor is not zero.

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REZIME

FRENETOVE FORMULE RIMAN-OTSUKIJEVOG PROSTORA

U radu su date Frenetove formule s obzirom na razne kovarijante diferencijale primenjene na kovarijante, odnosno na kontravarijantne indekse vektora.

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SOME FURTHER PROPERTIES OF TOTAL AND RICCI CURVATURE IN THE SELF-RECURRENT WEYL-OTSUKI SPACE OF THE SECOND KIND

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ABSTRACT

This note treats some problems concerning the concircularly semi-symmetric metric connection on a Riemannian manifold. Actually, if the main tensor of connection in the Weyl-Otsuki space of the second kind (SW-On) is self-recurrent in the adjoint Riemannian space, the metric covariant connection is semi-symmetric.

1. INTRODUCTION

This research work is mainly based on the results of three relatively fresh papers on the same problem (concircularly semi-symmetric metric connection, π-semi-symmetric metric connection) and in our last paper, published this year, applying the results of [7] on SW-On satisfying $\overset{\circ}{\nabla}_k p^i_j = \pi_k \delta^i_j$, now named the self-recurrent SW-On.

The π-semi-symmetric metric connection is defined in the following way:

$$T(X,Y) = \pi(Y)X - \pi(X)Y \tag{1.1}$$

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where π is an 1-form, for the torsion tensor

(1.2) $$\bar{\nabla} g = 0$$

for the metric tensor

(1.3) $$(\bar{\nabla}_X \pi)(Y) = 0,$$

for arbitrary vector fields X,Y.

From (1.1), (1.2), (1.3) this yields

(1.4) $$\bar{\nabla}_X Y = \nabla_X Y + \pi(Y)X - g(X,Y)P$$

where ∇ denotes the Levi-Civita covariant differentiation on the manifold, and P is such a vector field that $g(X,P) = \pi(X)$ for any vector field X.

The same is the meaning of the definition of concircularly semi-symmetric metric connection, but it is expressed in local coordinates (more familiar to us):

(1.5) $$\Gamma^i_{jk} = \{^i_{jk}\} + \delta^i_k p_i - g_{jk} p^i$$

(1.6) $$p_{j,k} = \nabla_k p_j = p_j p_k + \rho g_{jk}$$

where $\{^i_{jk}\}$ denotes the coefficients of the Levi-Civita connection on the Riemannian manifold, ∇ denotes the covariant differentiation regarding to $\{^i_{jk}\}$ and ρ is an arbitrary scalar function.

The main results regarding the concircularly semi-symmetric metric connection (π-semi-symmetric metric connection) from [1], [2], [7] are the following propositions:

Proposition 1. *The concircular curvature tensor of the concircularly semi-symmetric metric connection is equal to the concircular curvature tensor of the Levi-Civita connection.*

Proposition 2. *The projective curvature tensor of the concircularly semi-symmetric metric connection is equal to*

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the projective curvature tensor of the Levi-Civita connection.

Proposition 3. *The conformal curvature tensor of the concirculyrly semi-symmetric metric connection is equal to the conformal curvature tensor of the Levi-Civita connection.*

Proposition 4. *If a Riemannian manifold admits a semi-symmetric metric* π*-connection,* $\bar{\nabla}$*, then the manifold is of constant curvature if and only if it is of constant curvature for* $\bar{\nabla}$.

Proposition 5. *If a Riemannian manifold admits a semi-symmetric metric* π*-connection* $\bar{\nabla}$*, then the manifold is an Einstein manifold if and only if it is an Einstein manifold for the connection* $\bar{\nabla}$.

Proposition 6. *If a Riemannian manifold admits a semi-symmetric metric* π*-connection* $\bar{\nabla}$*, then a necessary and sufficient condition for the Ricci tensor to vanish is either the Levi-Civita curvature tensor being equal to the projective curvature tensor or to the conformal curvature tensor of the Levi-Civita connection.*

Proposition 7. *If a Riemannian manifold admits a semi-symmetric metric connection (*π*-connection) with a covariantly constant torsion tensor, then the manifold is Ricci-symmetric if and only if* $S(X,Y) = -(n-1)\pi(P)g(X,Y)$ $(R_{ij} = -(n-1) \cdot \pi^a \pi_a g_{ij})$.

Proposition 8. *The Ricci tensor of the semi-symmetric metric* π*-connection is symmetric if and only if the 1-form* π *is closed.*

Proposition 9. *If a Riemannian manifold of dimension* n $(n \geq 3)$ *admits a semi-symmetric metric* π*-connection* $\bar{\nabla}$ *with*

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P a Killing vector field, then a necessary and sufficient condition for the Ricci-tensor of $\bar{\nabla}$ *to be skew-symmetric is that the Ricci tensor of the Levi-Civita connection is given by:*

$$S(X,Y) = (n-2)\,\pi(P)g(X,Y) - \pi(X)\pi(Y)$$

(1.7)

$$[R_{ij} = (n-2)[\pi^a \pi_a g_{ij} - \pi_i \pi_j]]$$

Proposition 10. *If a Riemannian manifold of dimension* n (n ⩾ 3) *admits a semi-symmetric metric* π*-connection whose Ricci tensor is skew symmetric and if* P *is a Killing vector field, then the manifold cannot be Ricci-flat. If* n = 2 *and* P *is a Killing vector field, the Ricci tensor of* $\bar{\nabla}$ *is skew-symmetric if and only if the manifold is Ricci-flat.*

It has to be noticed that, in a local sense, π^i denote the contravariant components of π, or components of vector P.

We shall apply most of these results to the Weyl-Otsuki space of the second kind (denoted by SW-On). The covariant differentiation in such a space is given by (for example)

$$T^i_{jk,h} = P^i_a\, T^a_{bc|h}\, P^b_j\, P^c_k\,, \tag{1.8}$$

where P^i_j are coefficients of a linear isomorphism of the space, and $T^a_{bc|h}$ is the basic covariant differentiation given by two classical affine connections; one of them ($''\Gamma$) works exceptionally on covariant indices, and the other one ($'\Gamma$) works on contravariant ones. SW-On are metric spaces and there hold the following conditions:

a) $g_{ij,k} = \gamma_k m_{ij}$ (γ_k is a vector field and m_{ij} is a symmetric tensor field)

b) the connection $'\Gamma$ is symmetric

c) $P_{ij} = g_{ia}\, P^a_j$ is symmetric.

The inverse of tensor P is denoted by Q. The underlying Riemannian space is called the adjoint Riemannian space;

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its Levi-Civita covariant differentiation is denoted by $\mathring{\nabla}$. For further properties, the reader is urged to see, for example [3], [4], [5], [6].

If SW-On satisfies the condition

$$\mathring{\nabla}_k P^i_j = \pi_k \delta^i_j, \tag{1.9}$$

it is said to be self-recurrent.

2.

Let us consider a self-recurrent SW-On and its connection $\overset{''}{\Gamma}{}^m_{jk}$ (that is, the covariant part of the metric connection if γ_k, from condition a) vanishes). We can calculate its coefficients:

$$\overset{''}{\Gamma}{}^m_{jk} = \{^i_{jk}\} + \tilde{\pi}_j \delta^i_k - g_{jk}\tilde{\pi}^i, \tag{2.1}$$

where $\tilde{\pi}_j$ means the image of π_j by the linear isomorphism Q. We can see that $\overset{''}{\Gamma}{}^m_{jk}$ is a semi-symmetric metric connection.

In [5] facts were investigated about the conformal curvature tensor of the connection $\overset{''}{\Gamma}{}^m_{jk}$ and about its equality to the conformal curvature tensor of the adjoint Riemannian space. Next, in [6] conditions were investigated for the vector field π_i to satisfy the concircularity condition (and for $\overset{''}{\Gamma}{}^m_{jk}$ to be concircularly semi-symmetric metric). That condition holds in the adjoint Riemannian space for the vector field π_i:

$$\mathring{\nabla}_k \pi_j = \tilde{\pi}_j \pi_k + \pi_j \tilde{\pi}_k + \rho P_{jk}, \tag{2.2}$$

where ρ is a scalar function. It is easy to show that, if it is properly chosen, π_i is a harmonic vector field. So, for a harmonic vector field π_i, we can apply Propositions 1. and 2. to the self-recurrent SW-On, as we have done in [6].

According to [8], we have:

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Corollary 1. *If the one dimensional Betti number of the manifold M is different from zero, the covariant metric connection of the self-recurrent* SW-On *can be concircularly semi-symmetric metric connection (semi-symmetric metric* π*-connection)([1]).*

Except for this, two helpful equalities hold:

$$\mathring{\nabla}_k \tilde{\pi}_j = Q_j^a \mathring{\nabla}_k \pi_a - \tilde{\tilde{\pi}}_j \pi_k$$

(2.3)

$$\mathring{\nabla}_k Q_j^i = -\pi_k Q_a^i Q_j^a.$$

According to Proposition 4, we have

Corollary 2. *If the one-dimensional Betti number of the manifold M is different from zero, and if* π_i *is a harmonic vector field, the metric connection* $\overset{m}{\Gamma}$ *is of constant curvature if and only if the adjoint Riemannian space is of constant curvature.*

Suppose that the adjoint Riemannian space is of constant curvature, then

$$R_{jk\ell}^i = k(g_{jk}\delta_\ell^i - g_{j\ell}\delta_k^i) \tag{2.5}$$

$$R_{jk} = (n-1)kg_{jk} \text{ or } R_k^j = (n-1)k\delta_k^j \tag{2.6}$$

$$R = n(n-1)k \tag{2.7}$$

In regard to Corollary 2, analogic relations hold for $\overset{m}{R}_{jk\ell}^i$, $\overset{m}{R}_{jk}$, $\overset{m}{R}$, $\overset{m}{K}$.

From [1], we have

$$\overset{m}{R}_{ij} = R_{ij} + (n-1)\tilde{\pi}^a \tilde{\pi}_a g_{ij}, \tag{2.8}$$

and

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(2.9) $$\overset{m}{''R} = R + n(n-1)\tilde{\pi}^a\tilde{\pi}_a.$$

According to (2.7), we have

$$n(n-1)\overset{m}{''K} = n(n-1)K + n(n-1)\tilde{\pi}^a\tilde{\pi}_a$$

and, consequently

Theorem 1. *If the vector* π_i *is harmonic and if the adjoint Riemannian space is of constant Gaussian curvature* K, *then the connection* $\overset{m}{''\Gamma}$ *also is of constant Gaussian curvature* $\overset{m}{''K} = K + \pi^a\pi_a$. *If their total curvatures are equal, the* SW-On *is trivial.*

After this, according to Proposition 5, we have

Proposition 11. *A self-recurrent* SW-On *with* π_i *a harmonic vector field is an Einstein space in regard to linear connection* $\overset{m}{''\Gamma}$ *if and only if the adjoint Riemannian space is an Einstein space.*

Moreover, taking into account (2.8), we can state

Proposition 12. *The Ricci tensor of* $\overset{m}{''\Gamma}$ *in a self-recurrent* SW-On *with* π_i *a harmonic vector field is a symmetric tensor if and only if the Ricci tensor of the adjoint Riemannian space is symmetric.*

3.

This section is concerned with the Ricci curvature of a self-recurrent SW-On with harmonic π_i in relation to the Ricci curvature of the adjoint Riemannian space.

Before all the calculations, we can state

Proposition 13. *The Ricci curvature of the adjoint*

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Riemannian space cannot be positive definite in the direction π_i.

Now, we shall consider the case when the Ricci tensor of $\overset{m}{\Gamma}$ vanishes. In this case, we get

$$S(x,y) = -(n-1)\pi(P)g(x,y)$$

$$(R_{ij} = -(n-1)\tilde{\pi}^a\tilde{\pi}_a g_{ij}) \tag{3.1}$$

and

$$R = -n(n-1)\tilde{\pi}^a\tilde{\pi}_a \tag{3.2}$$

Consequently, the adjoint Riemannian space is an Einstein space. Then, according to Proposition 6, we have either

$$\delta^i_k R_{j\ell} = \delta^i_j R_{k\ell} \tag{3.3}$$

or

$$R^i_k g_{\ell j} - R^i_j g_{\ell k} + R_{\ell j}\delta^i_k - R_{\ell k}\delta^i_j = \frac{R}{n-1}(\delta^i_k g_{\ell j} - \delta^i_j g_{\ell k}) \tag{3.4}$$

So, we have:

Theorem 2. *The necessary and sufficient condition for the Ricci tensor of the connection* $\overset{m}{\Gamma}$ *to vanish is either (3.3) or (3.4), for the Ricci tensor and the scalar curvature of the adjoint Riemannian space.*

If the Ricci tensor of the connection $\overset{m}{\Gamma}$ vanishes, so does its scalar curvature. So, we have for this case:

Theorem 3. *The curvature tensor of the connection* $\overset{m}{\Gamma}$ *is equal to the curvature tensor of the adjoint Riemannian space if and only if either (3.3) or (3.4) holds.*

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or

Corollary 3. *The relation*

$$(\tilde{\pi}_j\pi_i + \pi_j\tilde{\pi}_i + \rho P_{ij} + \frac{1}{2}g_{ij}\tilde{\pi}_a\tilde{\pi}^a)g_{k\ell} -$$

$$- (\tilde{\pi}_k\pi_i + \pi_k\tilde{\pi}_i + \rho P_{ki} + \frac{1}{2}g_{ki}\tilde{\pi}_a\tilde{\pi}^a)g_{j\ell} -$$

$$(3.5) \quad - (\tilde{\pi}_j\pi_\ell + \pi_j\tilde{\pi}_\ell + \rho P_{j\ell} + \frac{1}{2}g_{j\ell}\tilde{\pi}_a\tilde{\pi}^a)g_{ki} +$$

$$+ (\tilde{\pi}_k\pi_\ell + \pi_k\tilde{\pi}_\ell + \rho P_{k\ell} + \frac{1}{2}g_{k\ell}\tilde{\pi}_a\tilde{\pi}^a)g_{ji} =$$

$$= \pi_j\pi_i g_{k\ell} - \pi_k\pi_i g_{j\ell} - \pi_j\pi_\ell g_{ki} + \pi_k\pi_\ell g_{ij}$$

holds if and only if (3.3) or (3.4) hold.

By the fact of Proposition 8, we have

Corollary 4. *The necessary and sufficient condition for the Ricci tensor of the connection* $\overset{m}{\Gamma}$ *(metric semi-symmetric π-connection) to be symmetric is that the 1-form* $\tilde{\pi}$ *is closed.*

Now, we shall investigate the conditions for the vector $\tilde{\pi}$ to be a Killing vector field. For a Killing vector field, we have

$$(3.6) \qquad \nabla_j v_i + \nabla_i v_j = 0$$

and, consequently

$$(3.7) \qquad \nabla_i v^i = 0$$

We use relations (2.3) and (3.6) and get

$$(3.7) \qquad \mathring{\nabla}_j\tilde{\pi}_i + \mathring{\nabla}_i\tilde{\pi}_j = \tilde{Q}_i^a\mathring{\nabla}_j\pi_a - \tilde{\tilde{\pi}}_i\pi_j + Q_j^a\mathring{\nabla}_i\pi_a - \tilde{\tilde{\pi}}_j\pi_i = 0$$

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which is the Killing equation for the vector $\tilde{\pi}_i = Q_i^a \pi_a$ and the Levi-Civita connection of the adjoint Riemannian space. But, the vector $\tilde{\pi}$ satisfies the concircularity condition. That fact is expressed by (2.2). After that, the Killing equation will take the form

$$\tilde{\pi}_i \tilde{\pi}_j = - \rho g_{ij} \tag{3.8}$$

where ρ is the scalar function appearing in the $\tilde{\pi}$-concircularity condition. We can express (3.8) in the following way:

$$\pi_i \pi_j = - \rho P_{ia} P_j^a = - \rho \overset{2}{P}_{ij} \tag{3.9}$$

(3.8) i.e. (3.9) holds if the vector field $\tilde{\pi}_i$ is a Killing vector field.

Now, if we apply (3.9) to (2.2), the following relation yields

$$\mathring{\nabla}_k \pi_j = - \rho P_{kj}.$$

As, consequently, we apply successive covariant differentiation to the components of the tensor P_{kj}, we get

$$\mathring{\nabla}_\ell \mathring{\nabla}_i P_{kj} = \mathring{\nabla}_\ell \pi_i g_{kj} = - P_{i\ell} g_{kj} \tag{3.10}$$

and, consequently

$$P_{sj} R^s_{ki\ell} + P_{ks} R^s_{ji\ell} = 0 \tag{3.11}$$

according to the Ricci identity for repeated covariant differentiation, and, hence

$$R^p_{ki\ell} = - Q^{pj} P_{ks} R^s_{ji\ell} \tag{3.12}$$

for the components of the curvature tensor of the adjoint Riemannian space.

So, we have

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Theorem 4. *If the vector $\tilde{\pi}_i$ is concircular and Killing simultaneously, the components of the curvature tensor in the adjoint Riemannian space have to satisfy condition (3.12).*

Corollary 5. *If the vector $\tilde{\pi}_i$ is concircular and Killing simultaneously, then*

$$Q^j_p P_{ks} R^s_{ji\ell} = - Q^j_k P_{ps} R^s_{ji\ell} \tag{3.13}$$

$$Q^j_p P_{ks} R^s_{ji\ell} = Q^j_i P_{\ell s} R^s_{jpk} \tag{3.13.a}$$

and

$$Q_j P_{ks} R^s_{ji\ell} = Q^{\ell j} P_{is} R^s_{jk\ell} \tag{3.13.b}$$

Applying the Ricci identity, we can also get the following relation:

$$R^s_{jk\ell}\pi_s = \rho_\ell P_{kj} + \rho\pi_\ell g_{kj} - \rho_k P_{\ell j} - \rho\pi_k g_{\ell j} \tag{3.14}$$

Now, we are concerned with the possibility of such a realization. We know that a concircularly semi-symmetric metric connection is realizable on a Riemannian manifold if the vector π_i is a harmonic vector field. Then

$$\rho = \frac{-2\tilde{\pi}^k \pi_k}{P^i_i} \tag{3.15}$$

and

$$0 = - \rho P^i_i = 2\tilde{\pi}^k \pi_k$$

i.e. the vectors π and $\tilde{\pi}$ are mutually orthogonal.

Theorem 5. *The vector $\tilde{\pi}$ can be concircular and Killing simultaneously if its image by the linear isomorphism P^i_j*

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is a harmonic vector field and if π *and* $\tilde{\pi}$ *are mutually orthogonal.*

Now, we can state first:

Corollary 6. *The Ricci curvature of the adjoint Riemannian space cannot be a negative definite along the direction* $\tilde{\pi}_i$, *if* $\tilde{\pi}$ *is orthogonal to* π.

Theorem 6. *For a harmonic vector field* π_i *and* $\tilde{\pi}_i$ *orthogonal to it, the Ricci tensor of* $\overset{m}{\Gamma}$ *is skew-symmetric if and only if the Ricci tensor of the Levi-Civita connection in the adjoint Riemannian space is given by*

$$R_{ij} = (n-2)(\tilde{\pi}^a \tilde{\pi}_a g_{ij} - \tilde{\pi}_i \tilde{\pi}_j) \tag{3.16}$$

or

$$Q^{\ell j} P_{ks} R^{s}_{ji\ell} = (2-n)(Q^{sa}\pi_a Q^{b}_{s} g_{ij} - Q^{b}_{i} Q^{a}_{j} \pi_b \pi_a). \tag{3.16.a}$$

Finally, we get

Theorem 7. *If the vector* π *satisfies condition (3.10) and if the Ricci tensor of* $\overset{m}{\Gamma}$ *is skew-symmetric, then* $\overset{m}{\Gamma}$ *cannot be Ricci-flat.*

Corollary 7. *If the vector* π *is harmonic and if* $\tilde{\pi}$ *is orthogonal to it, then the Ricci curvature of the adjoint Riemanian space is negatively definite along the direction of* π *and positively definite along the direction of* π *and positively definite along the direction of* $\tilde{\pi}$; *its one-dimensional Betti number cannot vanish and, except for this, it admits a one-parametric group of motions. If, moreover, we build a self-recurrent Weyl-Otsuki space of the second kind on such a Riemannian space, the connection* $\overset{m}{\Gamma}$ *cannot be Ricci flat if its Ricci tensor is skew-symmetric.*

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But, if the Ricci tensor of $\overset{m}{\Gamma}$ is skew-symmetric, the Gaussian curvature of $\overset{m}{\Gamma}$ vanishes. Then, according to Theorem 1. and Theorem 5. we have

Corollary 8. *For a harmonic vector field* π_i *and* $\tilde{\pi}_i$ *orthogonal to it, if the connection* $\overset{m}{\Gamma}$ *is of Gaussian curvature zero and if the Ricci tensor of the adjoint Riemannian space is given by the expression*

$$R_{ij} = (n-2)(\tilde{\pi}^a \tilde{\pi}_a g_{ij} - \pi_i \pi_j),$$

then the adjoint Riemannian space has a constant negative total curvature.

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REZIME

NEKE OSOBINE TOTALNE I RIČIJEVE KRIVINE SAMOREKURENTNOG VEJL-OTSUKIJEVOG PROSTORA DRUGE VRSTE

U radu su ispitane osobine totalne i Ričijeve krivine samorekurentnog Vejl-Otsukijevog prostora druge vrste (rad je baziran na rezultatima o koncirkularno-semisimetričnoj metričkoj koneksiji u Rimanovom prostoru. Ispituju se veze samorekurentnog SW-On sa pridruženim Rimanovim prostorom.

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ON INJECTIVE MODULES AND GENERALIZATIONS

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(Dedicated to Professor Yuzo Utumi on his sixtieth birthday)

ABSTRACT

In this paper two generalizations of injectivity are introduced and used to characterize some well-known classes of rings.

INTRODUCTION

Throughout, A denotes an associative ring with identity and A-modules are left unital, unless otherwise stated. J, Z stand respectively, for the Jacobson radical and left singular ideal of A. Two generalizations of injectivity, called CY and KY-injectivity, are introduced to study von Neumann regular and Noetherian rings. Conditions are given for two modules to have isomorphic injective hulls. This note contains the following results: (1) If M is a CY-injective module, then any cyclic submodule has an injective hull in M; (2) If A has a classical left quotient ring Q such that every

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divisible torsionfree A-module is CY-injective, then Q is semi-simple Artinian; (3) The following conditions are equivalent for a left non-singular ring A: (a) A is left Noetherian; (b) every CY-injective A-module is KY-injective; (c) every CY-injective A-module is injective; (4) For any left KY-injective ring A, A/J is von Neumann regular and J = Z; (5) If A is semi-primary, M, N A-modules such that either $r_M(Z)$ is isomorphic to $r_N(Z)$ or $r_M(J)$ is isomorphic to $r_N(J)$ (as left A-modules), then M and N have isomorphic injective hulls. It is also shown that in certain situations, proper direct summands of semi-prime right KY-injective rings possess non-trivial central idepotents.

For any A-module M, $Z(M) = \{z \in M/Lz = 0$ for some essential left ideal L of A$\}$ is the left singular submodule of M and M is called singular (resp. non-singular) if $Z(M) = M$ (resp. $Z(M) = 0$). Thus A is left non-singular iff $Z = 0$. An A-module M is called divisible if $M = cM$ for each non-zero-divisor c of A. M is called torsionfree if $cy \neq 0$ for every non-zero-divisor c of A and non-zero element y of M. Recall that A-module M is p-injective if, for any principal left ideal P of A, every left A-homomorphism of P into M extends to one of A into M. Then A is von Neumann regular iff every left (right) A-module is p-injective. Note that p-injective modules need not be flat and the converse is not true either. However, if I is a p-injective left ideal of A, then A/I is a flat A-module. If M is a maximal left ideal which is a two-sided ideal of A, then ${}_AA/M$ is flat iff A/M_A is injective iff A/M_A is p-injective. As usual, A is called a left V-ring if every simple left A-module is injective. Injective modules have been extensively studied by many authors since several years (cf. for example, [2], [3]). We now introduce the following two generalizations of injectivity, the first one being motivated by p-injectivity.

Definitions (1) *An A-module M is called CY-injective if, for any A-module Y, any cyclic submodule C of Y,*

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every left A-homomorphism of C into M extends to one of Y into M;

(2) *An A-module Y is called KY-injective if, for any complement submodule K of Y, any left A-monomorphism* $g : K \to Y$ *and left A-homomorphism* $F : K \to Y$, *there exists an endomorphism h of Y such that* $hg = f$.

It is easily seen that any direct summand of a KY-injective A-module is KY-injective.

CY-injectivity and KY-injectivity are distinct effective generalizations of injectivity. Recall that an A-module M is continuous if every submodule isomorphic to a complement submodule of M is a direct summand of M (cf. [6]). Continuous modules generalize quasiinjective modules. Since a continuous A-module is KY-injective, it follows that KY--injectivity does not imply CY-injectivity (cf. [1] and Proposition 8 below). The converse is not true either (cf. Theorem 5).

We start with various properties of CY-injective modules. Obviously, CY-injectivity implies p-injectivity but the converse is not true, as shown by our first proposition.

Proposition 1. *Let M be a CY-injective A-module. Then any cyclic submodule has an injective hull in M. In particular, every cyclic CY-injective A-module is injective.*

Proof. Let C be a cyclic submodule of M, E an injective hull of C. If g, j are the inclusion maps of C into M and C into E respectively, there exists a left A-homomorphism $h : E \to M$ such that $hj = g$. For any $d \in \ker h \cap C$, $d = g(d) = hj(d) = h(d) = 0$ and since C is an essential left submodule of E, then $\ker h = 0$ which implies that h is a monomorphism, whence $h(E)$ $(\approx E)$ is an injective A-module contained in M. This shows that C has an injective hull contained in M (because $C \subseteq h(E)$). In case M is cyclic, then it is obvious that M is injective.

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Corollary 1.1. (a) *A is a left V-ring iff every simple A-module is* CY*-injective;*

(b) *A is left self-injective regular iff every finitely generated left ideal of A is a* CY*-injective A-module.*

Recall that a ring Q is a classical left quotient ring of A if

(i) $A \subseteq Q$;

(ii) every non-zero-divisor of A is invertible in Q;

(iii) every element of Q is of the form $q = b^{-1}a$, $a, b \in A$, b being a non-zero-divisor.

As usual, a left (right) ideal of A is called reduced if it contains no non-zero nilpotent element.

Remark 1. (a) Let A have a classical left quotient ring Q. Then Q is injective iff Q is CY-injective. Consequently, if A is left non-singular with ${}_AQ$ CY-injective, then Q is left self-injective regular and is the maximal left quotient ring of A;

(b) If A is left Noetherian such that each prime factor ring contains a non-zero CY-injective left ideal, then A is left Artinian;

(c) A is a division ring iff A is a prime ring containing a non-zero reduced CY-injective left ideal (cf. [8, Proposition 6]).

Proposition 2. *A direct sum of A-modules is* CY*-injective if and only if each direct summand is* CY*-injective.*

Proof. Given $M = \bigoplus_{i\in I} M_i$, where each M_i $(i \in I)$ is a CY-injective A-module, we prove that M is Cy-injective. Let N be an A-module, $c \in N$, $f : Ac \to M$ a left A-homomorphism, $y = f(c) = y_{i1} + y_{i2} + \ldots + y_{ip}$, $y_{ij} \in M_{ij}$, $1 \le j \le r$. If $p_j : M \to M_{ij}$ is the natural projection for each j, $1 \le j \le r$, then $p_j f : Ac \to M_{ij}$ and since M_{ij} is CY-injective, there

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exists a left A-homomorphism $h_j : N \to M_{ij}$ which extends $p_j f$. Define $h : N \to M$ by $h(u) = h_1(u) + \ldots + h_r(u)$ for all $u \in N$. Then $h(c) = p_1 f(c) + \ldots + p_r f(c) = y_{i1} + \ldots + y_{ir} = y$ which shows that h extends f to N. This proves that M is CY-injective. Conversely, using the natural injection and projection, it is easily seen that a direct summand of a CY-injective A-module is CY-injective.

It is well-known that A is left Noetherian iff any direct sum of injective A-modules is injective.

Corollary 2.1. *If every CY-injective A-module is injective, then A is left Noetherian. Consequently, A is a principal left ideal ring iff every finitely generated left ideal of A is principal and every CY-injective A-module is injective.*

Corollary 2.2. *The following conditions are equivalent:* (a) *A is a left Noetherian left V-ring whose quasi-injective and CY-injective modules are injective;*

(b) *An A-module is quasi-injective iff it is CY-injective. (cf. [2, Proposition 20.4B]).*

Remark 2. If A has non-zero left socle S, than S is a CY-injective A-module iff every minimal left ideal is injective. Therefore, A is simple Artinian iff A is prime with a non-zero socle which is a left and right CY-injective A-module.

Remark 3. If A is von Neumann regular, then every cyclic submodule of a projective CY-injective A-module is injective.

Proposition 3. *Let A have a classical left quotient ring Q. If every divisible torsionfree A-module is CY-injective, then Q is semi-simple Artinian.*

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Proof. Let $C = Qy$ be a cyclic Q-module. It is sufficient to prove that C is a direct summand of every Q-module M containing it. Then every cyclic Q-module will be injective and the proposition will follow from [4, Theorem 3.2]. Since C is a torsionfree divisible A-module and ${}_AAy$ is essential in ${}_AC$, then C is injective (cf. the proof of Proposition 1). Therefore $M = C \oplus P$ and since M is divisible, then so is P. For any $y \in P$, any $q \in Q$, $q = b^{-1}d$, $b, d \in A$, if $u = qy = b^{-1}w$, where $w = dy$, since $P = bP$, then $w = bv$ for some $v \in P$ and hence $bu = w = bv$ which implies $u = v \in P \subseteq M$, showing that P is a left Q-module. Thus ${}_QM = {}_QC \oplus {}_QP$ which proves that C is an injective left Q-module.

Corollary 3.1. *If A has a von Neumann regular clasical left quotient ring Q and every p-injective torsionfree A-moduls is CY-injective, then Q is semi-simple Artinian.*

As usual, A is called left duo if every left ideal is a two-sided ideal.

Corollary 3.2. *A left duo ring whose divisible torsionfree left modules are CY-injective possesses a classical left quotient ring which is a finite direct sum of division rings.*

Proposition 4. *Let A be left non-singular such that every direct sum of the injective hulls of cyclic singular A-modules is injective. Then the singular submodule of any CY-injective A-module is injective.*

Proof. Let M be a CY-injective A-module with $Z(M) \neq 0$. If $0 \neq z \in Z(M)$, then Az has an injective hull U contained in M by Proposition 1 and since A is left non-singular, we know that U must be contained in Z(M). Let E denote the set of the injective hulls of all cyclic singular A-modules contained in M. Then the set F of all independent

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families $\{N_j\}$ of elements of E is an inductive set and by Zorn's Lemma, F has a maximal member $\{N_i\}_{i\in I_0}$. Now $K = \oplus_{i\in I} N_i \subseteq Z(M)$ and ${}_AK$ is injective by hypothesis, which implies $Z(M) = K \oplus V$. If $0 \neq w \in V$, then Aw has an injective hull W contained in Z(M). Then $W \cap K = 0$ yields a member of F which strictly contains $\{N_i\}_{i\in I_0}$, contradicting its maximality in F. Thus $V = 0$ and $Z(M) = K$ is injective.

Applying [7, Proposition 4], we get

Corollary 4.1. *If A is a left Noetherian ring whose divisible singular modules are CY-injective, then A is left hereditary.*

The next result connects CY-injectivity with KY-injectivity.

Theorem 5. *The following conditions are equivalent for a left non-singular ring A:*

(1) A *is left Noetherian;*

(2) A *is of left finite Goldie dimension and every direct sum of the injective hulls of cyclic singular A-modules is injective;*

(3) *Every CY-injective A-module is KY-injective;*

(4) *Every CY-injective A-module is injective.*

Proof. Obviously, (1) implies (2).

Assume (2). Let M be a CY-injective A-module. By Proposition 4, $M = Z(M) \oplus Q$, where ${}_AZ(M)$ is injective and ${}_AQ$ is non-singular. Since the injective hull of a non-singular A-module is non-singular, by [5, Theorem 2.5], any direct sum of the injective hulls of cyclic non-singular A-modules is injective and non-singular. As in the proof of Proposition 4, it can be shown that ${}_AQ$ (which is also CY-injective) is a direct sum of injective hulls of cyclic non-singular left submodules, whence ${}_AQ$ is injective and (2) implies (3).

Assume (3). If ${}_AM$ is CY-injective, E the injective hull of ${}_AM$, set $S = {}_AM \oplus {}_AE$. Then ${}_AS$ is CY-injective (Propo-

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sition 2) and is therefore KY-injective by hypothesis. If $i : M \to S$, $p : S \to M$ are the natural injection and projection ($p\,i$ is therefore the identity map on M), $j : M \to E$, $u : E \to S$ the inclusion maps, then there exists a map $h : S \to S$ such that $huj = i$. With $phu = q$, we have a left A-homomorphism $q : E \to M$ such that $qj = phuj = pi$, the identity map on M, which proves that ${}_AM$ is a direct summand of ${}_AE$. Thus M = E and (3) implies (4).

(4) implies (1) by Corollary 2.1.

Using [2, Theorem 24.20], the next result may be similarly proved.

Proposition 6. *The following conditions are equivalent:*

(1) A *is quasi-Frobeniusean;*

(2) *The direct sum of an injective and a projective* A*-modules is* KY*-injective.*

Looking at Theorem 5, we may ask the following: when are KY-injective A-modules injective? The next KY-injective analogue of [2, Proposition 20.4B] holds.

Remark 4. The following conditions are equivalent:
(1) Every KY-injective A-module is injective;
(2) The direct sum of any two KY-injective A-modules is KY-injective.

Remark 5. (1) If A is of left finite Goldie dimension such that every divisible singular A-module is injective, then A is left hereditary, left Noetherian; (2) The following are equivalent: (a) An A-module is CY-injective iff it is KY-injective; (b) All CY-injective and KY-injective A-modules are injective; (3) If the sum of any two KY-injective A-modules is KY-injective then A is a left hereditary, left Noetherian, left V-ring.

Note that if ${}_AM$ is KY-injective and N is a left submodule of M, then ${}_AN$ is a direct summand of ${}_AM$ iff N is a complement left submodule of M which is KY-injective.

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A is called left KY-injective if ${}_AA$ is KY-injective. An element c of A is called left regular if $l(c) = 0$. Then c is a non-zero-divisor iff it is left and right regular. The next result extends [6, Lemma 4.1].

Proposition 7. *Let A be a left KY-injective ring. Then* (1) *Every left regular element is right invertible in A; consequently, every left (right) A-module is divisible;* (2) $Z = J$ *and* A/Z *is von Neumann regular.*

Proof. (1) Let $c \in A$ such that $l(c) = 0$. Since the map $g : A \to A$ given by $g(a) = ac$ $(a \in A)$ is a monomorphism, then with $i : A \to A$ the identity map, there exists a left A-homomorphism $h : A \to A$ such that $hg = i$, which yields $1 = hg(1) = h(c) = ch(1)$, showing that any left regular element is right invertible. It follows that every non-zero-divisor is invertible and every left (right) A-module is divisible.

(2) For any $z \in Z$, $a \in A$, since $l(za) \cap l(1-za) = 0$, then $l(1-za) = 0$ which implies $1 - za$ right invertible in A by (1), showing that $z \in J$. In order to have $Z = J$, since $(J + Z)/Z$ is contained in the Jacobson radical of A/Z, it is sufficient to prove that A/Z is von Neumann regular. Let $0 \neq \bar{b} \in A/Z$, $\bar{b} = b + Z$, $b \in A$, $b \notin Z$. There exists a non-zero complement left ideal C of A such that $E = C \oplus l(b)$ is an essential left ideal. For any $0 \neq c \in C$, $cb \neq 0$ and since the map $g : C \to A$ given by $g(c) = cb (c \in C)$ is a monomorphism, if $i : C \to A$ is the inclusion map, then there exists a left A-homomorphism $h : A \to A$ such that $hg = i$. For every $c \in C$, $c = hg(c) = h(cb) = cbd$, where $d = h(1)$. Therefore $C \subseteq l(b-bdb)$ which yields $E \subseteq l(b-bdb)$, whence $\bar{b} = \bar{b}\bar{d}\bar{b}$ in A/Z, proving that A/Z is von Neumann regular.

Corollary 7.1. *A is left continuous regular iff A is a left non-singular left KY-injective ring whose complement left ideals are finitely generated.*

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Corollary 7.2. *A left Noetherian left KY-injective ring is left Artinian.*

Question: When is a left Noetherian left KY-injective ring quasi-Frobeniusean?

T is called a strongly regular ideal of A if T is a reduced two-sided ideal which is a regular ring. If A is left KY-injective, then any reduced principal left ideal is generated by an idempotent (cf. the proof of Proposition 7 (2)). If A is also semi-prime, then any reduced left ideal is a two-sided ideal of A which is a strongly regular ring. (We shall later look into conditions when non-reduced left ideals in semi-prime left KY-injective rings contain central idempotents.)

Remark 6. Let A be semi-prime left KY-injective. Then S, the sum of all reduced left ideals of A, coincides with the sum of all reduced two-sided ideals of A and is the unique maximal strongly regular ideal of A. If, further, every complement left ideal of A is finitely generated, then $A = S \oplus T$, where S is a left and right continuous strongly regular ring and T contains all the nilpotent elements of A.

We are now in a position to mention a few characteristic properties of semi-simple Artinian rings.

Combining [4, Theorem 3.2], Propositions 1 and 7, together with the proof of Theorem 5, we get

Proposition 8. *The following conditions are equivalent:*

(1) A *is semi-simple Artinian;*

(2) *Every cyclic semi-simple* A*-module is* CY*-injective;*

(3) *Every cyclic torsionfree* A*-module is* CY*-injective;*

(4) *Every finitely generated torsionfree* A*-module is* KY*-injective:*

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(5) A *is semi-prime left* KY*-injective satisfying the maximum condition on left annihilators.*

Recall that A is directly finite iff ${}_AA \oplus {}_AM \approx {}_AA$ implies $M = 0$.

Corollary 8.1. *Let* A *be directly finite such that any cyclic torsionfree* A*-module not isomorphic to* ${}_AA$ *is* CY*-injective. Then* A *is either semi-simple Artinian or an integral domain.*

Proof. It is clear that every principal left ideal of A is projective. Suppose that A is not a domain. Then there exists $b \in A$ such that $l(b) = Ae$, where e is a non-trivial idempotent. Since $A = Ae \oplus A(1-e)$ is directly finite, then both Ae and A(1-e) must be CY-injective and hence injective by Proposition 1. A is therefore left self-injective which implies that every cyclic torsionfree A-module is CY-injective and the corollary follows from Proposition 8.

The next remark also holds.

Remark 7. The following conditions are equivalent: (1) A is either semi-simple Artinian or a left principal ideal domain; (2) A is a directly finite ring such that any left ideal not isomorphic to ${}_AA$ is injective.

We now turn to conditions which will ensure that two injective modules are isomorphic. For any left A-module M, any two-sided ideal T of A, $r_M(T) = \{y \in M / Ty = 0\}$ is a left submodule of M. If M, N are A-modules and $f : M \to N$ is a left A-homomorphism, then $f(r_M(J)) \subseteq r_N(J)$.

Theorem 9. *Let* A *be a left* KY*-injective ring satisfying the maximum condition on left annihilators,* M, N *A-modules,* $u : M \to N$, $v : N \to M$ *left* A*-monomorphisms. If* E, H *are injective* A*-modules and* $f : M \to E$, $g : N \to H$ *are left* A*-monomorphisms such that* $f(r_M(J))$ (resp. $g(r_N(J))$ *is an essential left submodule of* $r_E(J)$ *(resp.* $r_H(J)$*), then* ${}_AE \approx {}_AH$.

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Proof. Since ${}_AH$ is injective, there exists a left A-homomorphism $h : E \to H$ such that $gu = hf$. If $y \in \ker h \cap \cap \mathrm{Im} f$, then $y = f(z)$, $z \in M$ and $g(u(z)) = h(f(z)) = h(y) = 0$ implies $u(z) = 0$, whence $z = 0$, yielding $y = 0$. Now $\ker h \cap \cap f(r_M(J)) = 0$ implies $\ker h \cap r_E(J) = 0$ (because $f(r_M(J))$ is essential in $r_E(J)$ (as left A-modules)). First suppose that $J \neq 0$. Since A is left KY-injective satisfying the maximum condition on left annihilators; then by Proposition 7, $J = Z$ is nilpotent. Let n be the least positive integer such that $J^n = 0$. Then $J^n \ker h = 0$ implies $J^{n-1} \ker h \subseteq r_E(J) \cap \cap \ker h = 0$. If $n-1 > 1$, we similarly have $J^{n-2} \ker h = 0$ and so on. Finally, we reach $\ker h = 0$. By symmetry, we also get a left A-monomorphism of H into E. It follows from [3, Theorem 1.13] that ${}_AE \approx {}_AH$. Now if $J = 0$, then A is semi-simple Artinian by Proposition 8 and in that case, ${}_AE \approx {}_AM \approx {}_AN \approx {}_AH$.

The next proposition may be similarly proved.

Proposition 10. (1) *If* A *is semi-primary,* M, N A-*-modules such that either* $r_M(Z)$ *is isomorphic to* $r_N(Z)$ *or* $r_M(J)$ *is isomorphic to* $r_M(J)$ *(as left* A*-modules) then* ${}_AM$ *and* ${}_AN$ *have isomorphic injective hulls;*

(2) *Let* A *satisfy the maximum condition on left annihilators. If* P, Q *are* A*-modules such that* $r_P(Z)$ *and* $r_Q(Z)$ *are isomorphic (as left* A*-modules), then* ${}_AP$ *and* ${}_AQ$ *have isomorphic injective hulls.*

Note that, in general, for any non-singular A-module M, $r_M(Z) = M$.

Proposition 11. *Let* A *be a commutative ring,* M *a non-singular injective* A*-module. For any ideal* T *of* A, $r_M(T)$ *is an injective submodule of* M.

Proof. Let E be an injective hull of $r_M(T)$ in M. For any $y \in E$, there exists an essential ideal L of a such that $Ly \subseteq r_M(T)$ which implies $LTy = TLy = 0$, whence Ty is contained in the singular submodule of M which is zero. The-

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refore $y \in r_M(T)$ which proves that $r_M(T)$ is an injective A-module.

Corollary 11. *Let A be a commutative non-singular ring satisfying the maximum condition on annihilators. For any non-singular divisible A-module M and any ideal T of A, $r_M(T)$ is an injective A-module (cf. [3, P. 102 ex. 18]).*

The proof of Theorem 9 and Proposition 11 yield

Remark 8. Let A be commutative with a nilpotent ideal U. If M is a submodule of a non-singular A-module N, then M is essential in N iff $r_M(U)$ is essential in $r_N(U)$.

Proposition 12. *The following conditions are equivalent for a commutative ring A:*

(1) *A is self-injective regular;*

(2) *For any finitely generated A-module M and any ideal P of A, $r_{M/Z(M)}(P)$ is an injective projective A-module;*

(3) *For any finitely generated A-module M and any ideal P of A, $r_{M/Z(M)}(P)$ is a CY-injective projective A-module.*

Proof. Assume (1). For any finitely generated A-module M, we know that $N = M/Z(M)$ is a non-singular A-module which is therefore injective and projective by [9, Corollary 6]. If P is an ideal of A, by Proposition 11, $r_N(P)$ is an injective submodule which is therefore a direct summand of N. Thus (1) implies (2).

(2) implies (3) evidently.

Assume (3). In as much as ${}_AA/Z$ is projective, we get $Z = 0$ and hence (3) implies (1) by Propositions 1 and 7.

A theorem of M. Ikeda - T. Nakayama asserts that if A is left self-injective, then for any left ideals L, S of A, $r(L \cap S) = r(L) + r(S)$. We now consider situations where certain proper direct summands of A contain non-trivial cen-

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tral idempotents.

Theorem 13. *Let A be a semi-prime left KY-injective ring such that* $r(L \cap S) = r(L) + r(S)$ *for any left ideals L, S. Let I be a non-singular CY-injective left ideal of A containing two non-zero principal left ideals P, Q with the following properties: P contains no direct sum of a pair of mutually isomorphic non-zero left ideals of A while Q contains no left ideal isomorphic to P. If* $A = Q \oplus K$, *then K contains a non-trivial central idempotent.*

Proof. For any $b \in I$, there exist $a \in A$ such that $b = bab$ (cf. the proof of Proposition 7). Consequently, P and Q are direct summands of ${}_AA$. By Zorn's Lemma, the set E of all left A-monomorphisms from some submodule of ${}_AP$ into ${}_AQ$ contains a maximal member g. Let $g : D \to Q$, $D \subseteq P$. Since P, Q are also direct summands of I, they are CY-injective a and hence are injective A-modules by Proposition 1. If $g(D) = F$, let $\hat{D}$, $\hat{F}$ be the injective hulls of D, F in P, Q respectively. If we suppose that $D \neq \hat{D}$, then g extends to a left A-homomorphism $\hat{g} : \hat{D} \to Q$. Since D is essential in $\hat{D}$, then $\hat{g}$ is a monomorphism belonging to E, which contradicts the maximality of g. This proves that $D = \hat{D}$ and therefore $F = \hat{F}$. If $P = D \oplus Az$, $z = z^2 \in I$, then $z \neq 0$ (in as much as Q contains no left ideal isomorphic to P). If $Q = F \oplus T$, $F = Au$, $T = Av$, u, v being idepotents in I, then we claim that $zAu = 0$. If not, let $0 \neq w = zdu$, $d \in A$. Since $l(w) = Ak$, $k = k^2$, the map $H : Az \to Aw$ given by $H(az) = aw$ for all $a \in A$ yields $Az/\ker H \approx Aw$. Then $\ker H = Akz$ implies $Az/Akz \approx Aw$. Since $Akz = As$, $s = s^2 \in I$, and $A = As \oplus A(1-s)$, then $Az = As \oplus Az(1-s)$, where $Az(1-s) = A(1-s) \cap Az$. Now $r = z(1-s) \in Az$, which implies $Ar = Ae$, $e = e^2 \in Az$, whence $Ae \approx Ac$, where $Aw = Ac$, $c = c^2 \in Au$, yielding $g^{-1}(Ac) \oplus Ae \subseteq P$ and $g^{-1}(Ac) \approx Ae$, thus contradicting the hypothesis on P. This proves that $zAu = 0$. If we suppose that $zAv \neq 0$, then there exist similarly non-zero idempotents $t \in Av$, $q \in Az$ and an isomorphism m of Aq onto At. Now the map $n : D \oplus Aq \to F \oplus At$ given

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by $n(p + aq) = g(p) + m(aq)$ for all $p \in D$, $a \in A$, is a monomorphism which contradicts the maximality of g in E. Therefore $zAv = 0$ also which yields $zQ = 0$, $z \neq 0$. Since Q is a direct summand of ${}_AA$, if $A = Q \oplus K$, then K is the left annihilator of h, where $Q = Ah$, $h = h^2$. If $T = QA$, in as much as A is semi-prime, $l(T) = r(T)$, $l(T) \cap T = 0$ and hence $A = r(l(T)) + r(T) = l(l(T)) + l(T)$ which implies that $A = l(l(T)) \oplus l(T)$. Again, since A is semi-prime, $l(Q) = l(T)$ is generated by a central idempotent and it follows that $K = l(h)$ contains a non-trivial central idempotent of A.

Corollary 13. *Let A be a semi-prime self-injective ring with non-zero socle. If Q is a non-zero non-singular injective left ideal which contains no minimal left ideal and* $A = Q \oplus V$, *then V contains a non-trivial central idempotent.*

We conclude with a last remark.

Remark 9. Let A be a left continuous regular ring, P, Q two non-zero injective left ideals having the same properties as in Theorem 13. If $A = Q \oplus K$, then K contains a non-trivial central idempotent.

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REZIME

O INJEKTIVNIM MODULIMA I UOPŠTENJIMA

U ovom radu dva uopštenja injektivnih modula su uvedena i korišćena za karakterizaciju nekih dobro poznatih klasa prstena.

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ON THE DISTRIBUTIVITY OF THE LATTICE OF L-VALUED SUBALGEBRAS OF FINITE ALGEBRAS

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ABSTRACT

The lattice $(S(A),\leq)$ of all L-valued (fuzzy) subalgebras of the given algebra A is considered. It is proved that for a finite algebra A, $(S(A),\leq)$ is isomorphic to a subdirect power of the lattice L, if $S(A)$ (the set of ordinary subalgebras of A) is closed under unions. Thereby, $(S(A),\leq)$ is distributive if and only if L is distributive. These results are applied to a class of groups.

1.

Let $A = (A,F)$ be a finite algebra, and $K \subseteq A$ a set of its constants. Let $(L,\wedge,\vee,0,1)$ be a bounded lattice (in the following denoted by L). An L-valued subset of (or: L-valued set on) A is a mapping $\overline{A} : A \to L$ (the set A and its subsets will be identified with their characteristic functions, where $0,1 \in L$; thus, $K(x) = 1$ if $x \in K$, othervise $K(x) = 0$). An L-valued subalgebra $\overline{B}$ of A is any L-valued subset of A, such that

a) $K \subseteq \overline{B}$, and

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b) $\overline{B}(f(x_1,\ldots,x_n)) \geq \overline{B}(x_1) \wedge \ldots \wedge \overline{B}(x_n)$, for all $x_1, \ldots, x_n \in A$, $f \in F_n \subseteq F$, $n \in N$.

$\overline{S(A)}$ is the set of all L-valued subalgebras of A, S(A) is the set of all ordinary subalgebras of A, and it is clear that $S(A) \subseteq \overline{S(A)}$.

$(\overline{S(A)},\leq)$ is a complete lattice (see, for example [4]).

Let $p \in L$, and $\overline{Ap} : A \to L$, where, for $x \in A$,

$$\overline{Ap}(x) = \begin{cases} 1, & \text{if } x \in K \\ p, & \text{otherwise.} \end{cases} \tag{1}$$

Lemma 1. *If* $p \in L$, *then* $\overline{Ap} \in \overline{S(A)}$.

Proof. Obvious.

Proposition 2. *A mapping* $f : L \to S(A)$, *given by* $f(p) = Ap$ *is an embedding of the lattice* L *into* $(S(A),\leq)$.

Proof. By the definition of $\overline{Ap}$, f is one-to-one, and

$$f(p \wedge q) = \overline{Ap \wedge q} = \overline{Ap \wedge Aq},$$

since

$$(\overline{Ap} \wedge \overline{Aq})(x) = \overline{Ap}(x) \wedge \overline{Aq}(x) = \begin{cases} 1, & \text{if } x \in K \\ p \wedge q, & \text{if } x \notin K. \end{cases}$$

In exactly the same way one can prove that

$$\overline{Ap \vee p} = \overline{Ap} \vee \overline{Aq}.$$

Proposition 3. *Let* $A = (A,F)$ *be a finite algebra with the property that* $B,C \in S(A)$ *implies* $B \cup C \in S(A)$. *Then the lattice* $(\overline{S(A)},\leq)$ *is isomorphic to a sublattice of* $I^{|A|-|K|}$, *where this sublattice is a subdirect power.*

Proof. Let $A = \{a_1,\ldots,a_n,c_1,\ldots,c_k\}$, $K = \{c_1,\ldots$

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$\ldots, c_k$, and let $f : \overline{S(A)} \to L^{|A|-|K|}$, where $\overline{B} \to \overline{S(A)}$, and for $i = 1, \ldots, n$

$$f(\overline{B})(i) = \overline{B}(a_i).$$

Since for $c \in K$, $\overline{B}(c) = 1$, it follows that f is one-to-one. In the following, we identify $f(\overline{B})(i)$ with $\overline{B}(a_i)$, i.e. we consider $\overline{S(A)}$ as a subset of $L^{|A|-|K|}$. It is closed under the lattice operations:

Let $\overline{B}, \overline{C} \in \overline{S(A)}$, and $x \in A \setminus K$. Then,

$$(B \sim C)(x) = B(x) \vee C(x) = \bigvee_{p \in L} p \cdot B_p(x) \vee \bigvee_{p \in L} p \cdot C_p(x) =$$

$$= \bigvee_{p \in L} p \cdot (B_p(x) \vee C_p(x)) = \left(\bigcup_{p \in L} p \cdot (B_p \cup C_p) \right)(x),$$

by the definition of a L-valued union, and a decomposition property ([3]).

Since for every $p \in L$, $B_p \cup C_p \in S(A)$ (as it was required for A), it follows that $\bigcup_{p \in L} p \, (B_p \cup C_p) \in \overline{S(A)}$. Clearly, if $B_p \cup C_p = D_p \in \overline{S(A)}$, then

$$\bigcup_{p \in L} p \cdot D_p(x,y) \geq \bigcup_{p \in L} p \cdot (D_p(x) \wedge D_p(y)) =$$

$$= \bigvee_{p \in L} p \cdot D_p(x) \wedge \bigvee_{p \in L} p \cdot D_p(y).$$

Thus, $\overline{B} \cup \overline{C} = \overline{B} \vee \overline{C}$, and hence for every $x \in A$

$$(\overline{B} \vee \overline{C})(x) = \overline{B}(x) \vee \overline{C}(x).$$

Now, since $f(B \vee C) = f(B \cup C)$, it follows that for $i = 1, \ldots, n$

$$f(\overline{B} \vee \overline{C})(i) = f(\overline{B} \cup \overline{C})(i) = (\overline{B} \cup \overline{C})(a_i) =$$

$$= \overline{B}(a_i) \vee \overline{C}(a_i) = (f(\overline{B}) \vee f(\overline{C}))(i),$$

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and thus (since $\overline{B} \cup \overline{C} = \overline{B} \vee \overline{C}$), $f(\overline{B} \vee \overline{C}) = f(\overline{B}) \vee f(\overline{C})$.

The proof that $\overline{S(A)}$ is closed under intersections is straightforward.

Thus we have proved that $(\overline{S(A)},\leq)$ is a sublattice of the lattice $L^{|A|}$ $^{-|K|}$. Moreover, it is a subdirect power, since for every $p \in L$, there is $\overline{B} \in \overline{S(A)}$ such that for $x \in A$, $\overline{B}(x) = p$. Namely, $\overline{B} = \overline{Ap}$ (defined by (1)).

Corollary 4. *Let* $A = (A,F)$ *be an algebra satisfying the conditions of Proposition 3. Then,* $(\overline{S(A)},\leq)$ *is a distributive lattice if and only if* L *is distributive.*

Proof. Let $(\overline{S(A)},\leq)$ be a distributive lattice. Then, by Proposition 2, L is distributive, as well.

If L is distributive, then $(\overline{S(A)},\leq)$ is also distributive, since it is a subdirect power of L, by Proposition 3.

Remark. The lattice of ordinary subalgebras $(S(A),\leq)$ of an algebra satisfying the conditions of Proposition 3 is distributive, since it is a sublattice of a distributive lattice $P(A) = B_2^A$ (B_2 is a Boolean algebra 2). Thus the properties of B_2 (used in meta-language) determine mainly the corresponding properties of the lattice $(S(A),\leq)$. Considering L-valued structures, i.e. taking the lattice L instead of B_2, one can see that properties are not always preserved.

THE CASE OF GROUPS

The preceding considerations, when applied to some classes of groups, can be formulated in a more concrete form.

Proposition 5. *Let* $(G,\cdot)$ *be a cyclic group of order* p^k, *p-prime,* $k \in N$, *and let* L *be any bounded lattice. Then,*

$$(\overline{S(G)},\leq) \cong (\{(q_1,\ldots,q_k) \mid q_1 \leq \ldots \leq q_k, q_j \in L\},\leq).$$

Proof. As it is shown in [1], all generators of the

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group G have the same value in L. The same is with the generators of every subgroup of G. Moreover, if g is a generator of G, and $\overline{H}$ an L-valued subgroup of G ($\overline{H} \in \overline{S(G)}$), then

$$\overline{H}(g) \leq \overline{H}(g^t), \quad t \in N \quad \text{(see also [1])}.$$

There are k different subgroups, and thus k generators having different values in L. The proof now follows directly from Proposition 3.

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REZIME

O DISTRIBUTIVNOSTI MREŽE L-VREDNOSNIH PODALGEBRI KONAČNIH ALGEBRI

Razmatra se mreža svih L-vrednosnih (L je mreža sa 0 i 1) podalgebri date algebre. Dokazuje se da je za konačne algebre ta mreža izomorfna sa jednim poddirektnim stepenom od L (tačno odredjenog reda), pod uslovom da je skup običnih podalgebri te algebre zatvoren u odnosu na uniju. Dokazuje se da je mreža L-vrednosnih podalgebri distributivna ako i samo ako je L distributivna mreža. Za cikličke grupe reda p^k (p-prost, $k \in N$), daje se i konkretan opis te mreže.

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A_t^m- И A_t-3-КВАЗИГРУППЫ

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РЕЗЮМЕ

В [1] введены понятия A_t^3- и A_t^4-алгебр. В [2] дано такое определение A_t^m-квазигруппы (A_t^m-алгебры), что A_t^3- и A_t^4--алгебры оказываются ее частными случаями. В [3] доказано, что A_t^m-квазигруппы являются координатизациоными системами конечных регулярных плоскостей, в которых прямые ℓ удовлетворяют условию $|\ell| \geq 3$. В [4] введены A_t-квазигруппы, являющеся одним из обобщений A_t^m-квазигрупп, и показано, что A_t-квазигруппы являются координатизационными системами конечных TCL-геометрий в которых прямые ℓ удовлетворяют условию $|\ell| \geq 3$. (Если $(\mathfrak{T}, \mathfrak{L})$ TCL-геометрия и $|\ell| \geq 2$ для всех $\ell \in \mathfrak{L}$, то $\mathfrak{L}$ является разбиением Хартманиса типа 2 множества $\mathfrak{T}$ [4,6-7]). В настоящей работе определяются и расматриваются A_t^m- и A_t-3-квазигруппы. Показано, что каждой A_t-3-квазигруппе (A_t^m-3-квазигруппе) соответствует конечная $TCL^{(2)}$-геометия (регулярная $TCL^{(2)}$-геометия). Обратное утверждение имеет место, если для каждой линии ℓ, $|\ell| = m \in N \setminus \{1,2,3\}$, существует A_m^m-3-квазигруппа.

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* ☆ *

Очевидно имеет место: если $(\mathfrak{T},A)$, $|\mathfrak{T}| = t \in N$, 3-квазигруппа, то любая тройка попарно различных элементов $a, b, c \in \mathfrak{T}$ порождает 3-подквазигруппу $([a,b,c],A)$ 3-квазигруппы $(\mathfrak{T},A)$.

ОПРЕДЕЛЕНИЕ 1. A_t^m-3-квазигруппой назовем 3-квазигруппу $(\mathfrak{T},A)$, $|\mathfrak{T}| = t \in N \setminus \{1,2,3\}$, тогда и только тогда, когда имеет место:

A1 $(\forall a \in \mathfrak{T})(\forall b \in \mathfrak{T})A(a^{i-1},b,a^{3-i}) = b$ для каждого $i \in \{1,2,3\}$[1]; и

A2 $(\forall a \in \mathfrak{T})(\forall b \in \mathfrak{T})(\forall c \in \mathfrak{T})(a \neq b \wedge a \neq c \wedge b \neq c \Rightarrow \Rightarrow |[a,b,c]| = m \in N \setminus \{1,2,3\})$[2].

ПРИМЕЧАНИЕ 1. Очевидно имеет место: в любой 3-квазигруппе $([a,b,c],A)$, относящейся к A2 справедливо A1[3].

Непосредственным следствием аксиомы A1 является следующее:

УТВЕРЖДЕНИЕ 1. A_t^m- квазигруппа $(\mathfrak{T},A)$ является идемпотентной 3-квазигруппой, т.е. в $(\mathfrak{T},A)$ имеет место формула:

$$(\forall a \in \mathfrak{T})A(a^3) = a.$$

Из следующего утверждения происходит, что A_t^m-квазигруппа является обобщением A_t^m-квазигруппы:

---

1) Каждый $a \in \mathfrak{T}$ является единицей 3-квазигруппы $(\mathfrak{T},A)$.

2) На множестве $\mathfrak{T}$, удовлетворяющем условию $|\mathfrak{T}| \leq 3$ не существует 3-квазигруппа, удовлетворяющая условию A1.

3) Закон $A(x^{i-1},y,x^{3-i}) = y$ имеет место для любых $x,y \in \mathfrak{T}$.

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УТВЕРЖДЕНИЕ 1′. В 3-квазигруппе $(\mathfrak{T},A)$, $|\mathfrak{T}| = t \in N \setminus \{1,2,3\}$, справедливо A1 тогда и только тогда, когда имеет место:

$$\underline{A1'} \quad (\forall a_1 \in \mathfrak{T})(\forall a_2 \in \mathfrak{T})(\forall a_3 \in \mathfrak{T})(|\{a_1^3\}| \in \{1,2\} \Rightarrow \Rightarrow A(a_1^3) \in \{a_1^3\}).$$

ДОКАЗАТЕЛЬСТВО.

а) Непосредственно получаем, что имеет место следующая импликация:

$$A1 \Rightarrow A1'.$$

б) Пусть $(\mathfrak{T},A)$ 3-квазигруппа, удовлетворяющая условию A1′. Существуют только следующие возможности:

$$A(a^{i-1},b,a^{3-i}) = a$$

или

$$A(a^{i-1},b,a^{3-i}) = b$$

для каждого $i \in \{1,2,3\}$.

Если $a \neq b$[1], то первая возможность противоречит предположению, что $(\mathfrak{T},A)$ 3-квазигруппа. Именно из

$$A(a^{i-1},b,a^{3-i}) = a \wedge a \neq b$$

находим, что имеет место формула:

$$(\forall a \in \mathfrak{T})(\forall b \in \mathfrak{T})(A(a^{i-1},a,a^{3-i}) = A(a^{i-1},b,a^{3-i}) \wedge a \neq b).$$

Так как эта формула эквивалентна формуле

$$\neg(\forall a \in \mathfrak{T})(\forall b \in \mathfrak{T})(A(a^{i-1},a,a^{3-i}) = A(a^{i-1},b,a^{3-i}) \Rightarrow a = b),$$

утверждение доказано.

[1] $|\mathfrak{T}| \in N \setminus \{1,2,3\}$

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A1′ является частным случаем следующей формулы:

A1$_n$ $(\forall a_1 \in \mathfrak{T}) \ldots (\forall a_n \in \mathfrak{T})(|\{a_1^n\}| \in \{1,\ldots,n-1\} \Rightarrow$
$\Rightarrow A(a_1^n) \in \{a_1^n\})$.

Для n = 2, A1$_n$ становится следующей формулой:

$$(\forall a_1 \in \mathfrak{T})(\forall a_2 \in \mathfrak{T})(|\{a_1^2\}| \in \{1\} \Rightarrow A(a_1^2) \in \{a_1^2\}),$$

эквивалентной формуле:

$$(\forall x \in \mathfrak{T})A(x,x) = x.$$

УТВЕРЖДЕНИЕ 2. В A_t^m-3-квазигруппе $(\mathfrak{T},A)$ имеет место:

A3 Каждую 3-квазигруппу $([a,b,c],A)$ порождает любая тройка попарно различных элементов множества $[a,b,c]$.

ДОКАЗАТЕЛЬСТВО. Пусть $\bar{a},\bar{b},\bar{c}$ любые попарно различные элементы множества $[a,b,c]$. Так как $\bar{a},\bar{b},\bar{c}$ попарно различные, они порождают 3-подквазигруппу $([\bar{a},\bar{b},\bar{c}],A)$, обладающую свойством A1[1] такую, что имеет место:

(a) $$|[\bar{a},\bar{b},\bar{c}]| = m$$ [2].

Учитывая, что

$$\bar{a},\bar{b},\bar{c} \in [a,b,c],$$

находим, что

$$[\bar{a},\bar{b},\bar{c}] \subseteq [a,b,c].$$

Отсюда, учитывая (a) и A2, находим, что

---

1) Примечание 1.

2) A2.

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$$[\bar{a},\bar{b},\bar{c}] = [a,b,c].$$

Так как a,b,c любые попарно различные элементы множества [a,b,c], утверждение доказано.

УТВЕРЖДЕНИЕ 3. В A_t^m-3-квазигруппе имеет место:

A4 $[\bar{a},\bar{b},\bar{c}] \neq [a,b,c] \Rightarrow |[\bar{a},\bar{b},\bar{c}] \cap [a,b,c]| \leq 2.$

ДОКАЗАТЕЛЬСТВО. Пусть $a,b,c,\bar{a},\bar{b},\bar{c}$ любые элементы, удовлетворяющие условию

(б) $$[\bar{a},\bar{b},\bar{c}] \neq [a,b,c].$$

Предположим,что имеет место:

(ц) $$|[a,b,c] \cap [a,b,c]| > 2$$ [1].

Из (ц) получаем, что существуют $p,q,r \in \mathfrak{T}$ такие, что:

$$p,q,r \in [a,b,c];$$

$$p,q,r \in [a,b,c]; \text{ и}$$

$$p \neq q,\ p \neq r,\ q \neq r.$$

Отсюда, учитывая утверждение 2, получаем, что спроведливы равенства

$$[\bar{a},\bar{b},\bar{c}] = [p,q,r] \quad \text{и} \quad [p,q,r] = [a,b,c],$$

т.е., что имеет место равенство

$$[\bar{a},\bar{b},\bar{c}] = [a,b,c].$$

Так как это равенство противоречит условию (б), находим,что

[1] $\Leftrightarrow \neg|[\bar{a},\bar{b},\bar{c}] \cap [a,b,c]| \leq 2.$

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утверждение доказано.

Подобными рассуждениями доказывается, что имеет место и следующее утверждение:

УТВЕРЖДЕНИЕ 4. Пусть $(\mathfrak{T},A)$, $|\mathfrak{T}| = t \in N \setminus \{1,2,3\}$, 3-квазигруппа удовлетворяющая условию A1. Тогда имеет место:

$$A3 \Leftrightarrow A4.$$

ОПРЕДЕЛЕНИЕ 2. A_t-3-квазигруппой назовем 3-квазигруппу $(\mathfrak{T},A)$, $|\mathfrak{T}| = t \in N \setminus \{1,2,3\}$, тогда и только тогда, когда имеет место A1 и A3[1].

Очевидно, что каждая A_t^m-квазигруппа является A_t-квазигруппой. Обратное не имеет место (утверждение 9).

A_4^4-3-квазигруппы и A_5^5-3-квазигруппы существуют: табл. 1_1- 1_4 и 2_1-2_5. Более того, имеет место:

УТВЕРЖДЕНИЕ 5. Если (Q,A) и $(\bar{Q},\bar{A})$ являются A_4^4-3-квазигруппами, то каждая биекция f множества Q на множество $\bar{Q}$ является изоморфизмом (Q,A) на $(\bar{Q},\bar{A})$.

ДОКАЗАТЕЛЬСТВО. Пусть $Q = \bar{Q} = \{1,2,3,4\}$. Пусть, далее, f подстановка множества $\{1,2,3,4\}$. Учитывая A1, находим, что имеют место следующие равенства:

$$fA(x,x,z) = fA(x,y,z) = fA(z,x,x) = fz$$

и

$$\bar{A}(fx,fx,fz) = \bar{A}(fx,fz,fx) = \bar{A}(fz,fx,fx) = fz$$

для любых $x,z \in \{1,2,3,4\}$. Далее, учитывая A2, находим, что имеет место: если $|\{x,y,z\}| = 3$ и $u \in \{1,2,3,4\} \setminus \{x,y,z\}$, то имеют место равенства:

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[1] A1 и A4 (в силу утверждения 4).

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A_1	1	2	3	4
1	1	2	3	4
2	2	1	4	3
3	3	4	1	2
4	4	3	2	1

Табл. 1_1

A_2	1	2	3	4
1	2	1	4	3
2	1	2	3	4
3	4	3	2	1
4	3	4	1	2

Табл. 1_2 $\mathfrak{T} = \{1,2,3,4\}$

A_3	1	2	3	4
1	3	4	1	2
2	4	3	2	1
3	1	2	3	4
4	2	1	4	3

Табл. 1_3

A_4	1	2	3	4
1	4	3	2	1
2	3	4	1	2
3	2	1	4	3
4	1	2	3	4

Табл. 1_4

A_1	1	2	3	4	5
1	1	2	3	4	5
2	2	1	5	3	4
3	3	4	1	5	2
4	4	5	2	1	3
5	5	3	4	2	1

Табл. 2_1

A_2	1	2	3	4	5
1	2	1	4	5	3
2	1	2	3	4	5
3	5	3	2	1	4
4	3	4	5	2	1
5	4	5	1	3	2

Табл. 2_2 $\mathfrak{T} = \{1,2,3,4,5\}$

A_3	1	2	3	4	5
1	3	5	1	2	4
2	4	3	2	5	1
3	1	2	3	4	5
4	5	1	4	3	2
5	2	4	5	1	3

Табл. 2_3

A_4	1	2	3	4	5
1	4	3	5	1	2
2	5	4	1	2	3
3	2	5	4	3	1
4	1	2	3	4	5
5	3	1	2	5	4

Табл. 2_4

A_5	1	2	3	4	5
1	5	4	2	3	1
2	3	5	4	1	2
3	4	1	5	2	3
4	2	3	1	5	4
5	1	2	3	4	5

Табл. 2_5

$$A(x,y,z) \overset{\text{деф}}{=\!=} A_x(y,z);$$

$$x,y,z \in \mathfrak{T}$$

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$$fA(x,y,z) = fu$$

и

$$\bar{A}(fx,fy,fz) = fu.$$

Утверждение доказано.

ПРИМЕЧАНИЕ 2. A_4^4-3-квазигруппа $(\{1,2,3,4\},A)$ определяет (тривиальную) систему четверок Штайнера $(\{1,2,3,4\}, \{\{1,2,3,4\}\})$, и обратно.

★ ☆ ☆ ★

В [4] "L-геометрия" имеет следующий смысл:

Пусть $\mathfrak{T}$ непустое множество[1] и пусть непустое множество $\mathfrak{L}$ множество некоторых непустых подмножеств множества $\mathfrak{T}$. Элементы множества $\mathfrak{T}$ называются точками, элементы множества $\mathfrak{L}$ прямыми. Объект $(\mathfrak{T},\mathfrak{L})$ называется L-геометрия тогда и только тогда, когда выполняются следующие условия:

L1 через любые две различные точки проходит не более одной прямой;

и

L2 каждая точка находится по меньшей мере в одной прямой.

Если в $(\mathfrak{T},\mathfrak{L})$ имеет место:

L1′ через любые две различные точки проходит одна и только одна прямая;

то в $(\mathfrak{T},\mathfrak{L})$ справедливы условия L1-L2.

L-геометрию $(\mathfrak{T},\mathfrak{L})$, в которой имеет место L1′, автор, в [4], позволил себе назвать TCL-геометрией[2].

---

1) В [4] речь идет о конечном множестве.

2) В самом деле, если $(\mathfrak{T},\mathfrak{L})$ является TCL-геометрией и $|\ell| \geq 2$, для всех $\ell \in \mathfrak{L}$, то $\mathfrak{L}$ является разбиением Хартманиса типа 2 множества $\mathfrak{T}$[6-7].

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Носители LN-к-полусетей являются L-геометриями [5]. L-геометрии являются носителями и некоторых RN-к-полусетей. Но, существуют RN-к-полусети, имеющие носители, не являющиеся L-геометриями. Именно в некоторых, кроме L2, имеет место:

L(2)1 через каждые три попарно различные точки проходит не более одной линии.

Объект $(\mathfrak{T},\mathfrak{L})$, в котором имеют место L(2)1 и L2, позволим себе назвать L(2)-геометрией. L(2)-геометрию $(\mathfrak{T},\mathfrak{L})$, в которой имеет место

L(2)1′ через каждые три попарно различные точки проходит одна и только одна линия;

назовем TCL(2)-геометрия[1]. TCL(2)-геометрию $(\mathfrak{T},\mathfrak{L})$ назовем регулярной тогда и только тогда, когда имеет место:

L3 $$(\forall \ell \in \mathfrak{L})(\forall \ell' \in \mathfrak{L})\ |\ell| = |\ell'|.$$

Пусть $(\mathfrak{T},A)$, $|\mathfrak{T}| = t \in N \setminus \{1,2,3\}$, A_t-3-квазигруппа. Пусть, далее,

$$\mathfrak{L} \overset{\text{def}}{=} \{[a,b,c] \mid a,b,c \in \mathfrak{T} \wedge a \neq b \wedge a \neq c \wedge b \neq c\}.$$

Если, при этом, элементы множества $\mathfrak{T}$ назовем точками а элементы множества $\mathfrak{L}$ линиями, то, учитывая определение 2 и утверждение 4, находим, что имеет место следующее утверждение:

УТВЕРЖДЕНИЕ 6. Если $(\mathfrak{T},A)$, $|\mathfrak{T}| = t \in N \setminus \{1,2,3\}$, A_t-3-квазигруппа, то $(\mathfrak{T},\mathfrak{L})$, где

$$\mathfrak{L} \overset{\text{def}}{=} \{[a,b,c] \mid a,b,c \in \mathfrak{T} \wedge a \neq b \wedge a \neq c \wedge b \neq c\},$$

---

[1] В самом деле, если $(\mathfrak{T},\mathfrak{L})$ является TCL(2)-геометрией и $|\ell| \geq 3$ для всех $\ell \in \mathfrak{L}$, то $\mathfrak{L}$ является разбиением Хартманиса типа 3 множества $\mathfrak{T}$[6-7].

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$TCL^{(2)}$-геометрия.

Отсюда, учитывая определение 1 и утверждение 2, находим, что имеет место следующее утверждение:

СЛЕДСТВИЕ 7. Если $(\mathfrak{T},A)$, $|\mathfrak{T}| = t \in N \setminus \{1,2,3\}$, A_t^m-3-квазигруппа, то $(\mathfrak{T},\mathfrak{L})$, где

$$\mathfrak{L} \overset{\text{деф}}{=\!=} \{[a,b,c] \mid a,b,c \in \mathfrak{T} \wedge a \neq b \wedge a \neq c \wedge b \neq c\},$$

регулярная $TCL^{(2)}$-геометрия.

УТВЕРЖДЕНИЕ 8. Пусть $(\mathfrak{T},\mathfrak{L})$, $|\mathfrak{T}| \in N$, $TCL^{(2)}$-геометрия такая, что имеет место:

а) для каждой $\ell \in \mathfrak{L}$ $|\ell| \in N \setminus \{1,2,3\}$;

$$|\ell| \in \{m_1,\ldots,m_p\},\ p \in N.$$

Тогда, если

б) для каждого $m \in \{m_1,\ldots,m_p\}$ существует A_m^m-3-квазигруппа;

то

в) существует 3-арна операция A на $\mathfrak{T}$ такая, что $(\mathfrak{T},A)$ является A_t-3-квазигруппой; $|\mathfrak{T}| = t$.

ДОКАЗАТЕЛЬСТВО. Пусть $(\mathfrak{T},\mathfrak{L})$ $TCL^{(2)}$-геометрия, удовлетворяющая условию а). Отсюда, учитывая условие б), находим, что имеет место:

1° на каждой линии $\ell_i \in \mathfrak{L}$, $i \in I$, можно определить тернарную операцию $A^{(i)}$, $i \in I$, такую, что $(\ell_i,A^{(i)})$, $i \in I$, становится A_m^m-3-квазигруппой, $m = |\ell_i|$.

Учитывая $L^{(2)}1'$ и 1°, находим, что имеет место:

2° для любых попарно различных $a,b,c \in \mathfrak{T}$ существует одна и только одна (из, в 1° выбранных) $A^{(i)}$, $i \in I$, такая, что $A^{(i)}(a,b,c) \in \ell_i \subseteq \mathfrak{T}$.

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Имея в виду $L^{(2)}1'$ и а), любые $a,b \in \mathfrak{T}$ являются элементами по меньшей мере одного множества ℓ_i, $i \in I$. Поэтому, имея в виду 1°, находим, что имеет место:

3° для любых $a,b \in \mathfrak{T}$ существует по меньшей мере одна (из, в 1°, выбранных) $A^{(i)}$, $i \in I$, такая, что

$$A^{(i)}(a,a,b) = A^{(i)}(a,b,a) = A^{(i)}(b,a,a) = b$$

для всех $i \in I$, удовлетворяющих условию $a,b \in \ell_i$.

На основании 2° и 3° находим, что имеет место:

4°

$$A \overset{\text{деф}}{=\!=} \bigcup_{i \in I} A^{(i)}$$

является тернарной операцией в множестве $\mathfrak{T}$.

На основании 3° находим, что имеет место:

5° В 3-группоиде $(\mathfrak{T},A)$ справедливо А1.

Далее, на основании $L^{(2)}1'$ и 1° находим, что имеет место:

6° В 3-группоиде $(\mathfrak{T},A)$ справедливо А4.

Пусть $a,b,c \in \mathfrak{T}$ любые элементы, удовлетворяющие условию: $a \neq b \wedge a \neq c \wedge b \neq c$. Отсюда, учитывая $L^{(2)}1'$ и 1°, находим что $([a,b,c],A)$ является A_m^m-3-квазигруппой. Поэтому каждое из уравнений

$$A(x,a,b) = c, \quad A(a,y,b) = c, \quad A(a,b,z) = c$$

обладает одним и только одним решением, принадлежащим множеству $[a,b,c]$. Предположение, что существуют и решения, не принадлежащие множеству $[a,b,c]$, противоречит условию А4 (6°). Далее, учитывая 4° и 5°, находим, что и уравнения

$$A(x,a,a) = b, \quad A(a,y,a) = b, \quad A(a,a,z) = b$$

обладают одним и только одним решением. Отсюда находим, что

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имеет место:

7° 3-группоид $(\mathfrak{T}, A)$ является 3-квазигруппой.

Наконец, учитывая 5°, 6° и 7°, находим, что утверждение доказано.

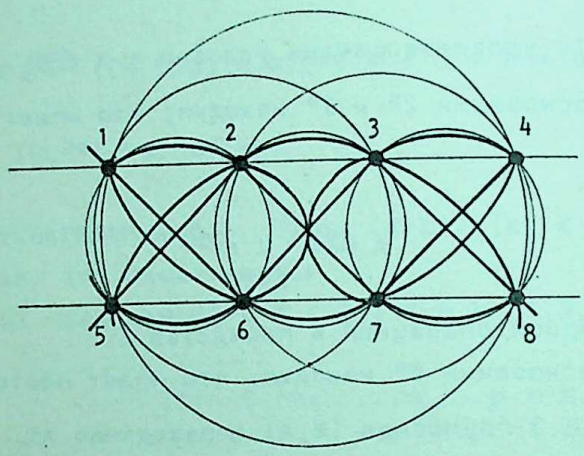

Рис. 1

На рис. 1 изображена регулярная $TCL^{(2)}$-геометрия $(\mathfrak{T}, \mathfrak{L})$, где:

$\mathfrak{T} = \{1,\ldots,8\}$;

$\ell_1 = \{1,2,3,4\}$, $\ell_2 = \{5,6,7,8\}$,

$\ell_3 = \{1,2,5,6\}$, $\ell_4 = \{2,3,6,7\}$,

$\ell_5 = \{3,4,7,8\}$, $\ell_6 = \{1,3,5,7\}$,

$\ell_7 = \{1,4,5,8\}$, $\ell_8 = \{2,4,6,8\}$,

$\ell_9 = \{1,3,6,8\}$, $\ell_{10} = \{2,4,5,7\}$,

$\ell_{11} = \{1,2,7,8\}$, $\ell_{12} = \{3,4,5,6\}$,

$\ell_{13} = \{2,3,5,8\}$, $\ell_{14} = \{1,4,6,7\}$.

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Отсюда, учитывая факт, что A_4^4-3-квазигруппы существуют (табл. 1_1-1_4), на основании утверждения 8, находим, что имеет место: A_8^4-3-квазигруппы существуют.

На рис. 2_1 изображена 4-сеть $(\mathfrak{T}, L_1, L_2, L_3, L_4)$ порядка 3[1]. Ее носитель $(\mathfrak{T}, \mathfrak{L})$, $\mathfrak{L} = L_1 \cup L_2 \cup L_3 \cup L_4$, удовлетворяет условиям: $|\mathfrak{T}| = 9$ и $|\mathfrak{l}| = 3$ для всех $\mathfrak{l} \in \mathfrak{L}$. На рис. 2_2 изображена $L^{(2)}$-геометрия $(\bar{\mathfrak{T}}, \bar{\mathfrak{L}})$, определенная следующим образом:

$$\bar{\mathfrak{T}} \overset{\text{деф}}{=\!=} \mathfrak{T} \cup \{10\},\ 10 \notin \mathfrak{T}, \quad \text{и } \bar{\mathfrak{l}} \in \bar{\mathfrak{L}} \overset{\text{деф}}{=\!=} \bar{\mathfrak{l}} \setminus \{10\} \in \mathfrak{L}.$$

Имеет место:

1) через каждые три попарно различные точки $a,b,10 \in \bar{\mathfrak{T}}$ проходит одна и только одна линия $\bar{\mathfrak{l}} \in \bar{\mathfrak{L}}$; и

2) если $a,b,c \in \bar{\mathfrak{T}} \setminus \{10\}$[2] и $|\{a,b,c\}| = 3$, то не существует $\bar{\mathfrak{l}} \in \bar{\mathfrak{L}}$ такая, что имеет место: $a \in \bar{\mathfrak{l}} \wedge b \in \bar{\mathfrak{l}} \wedge c \in \bar{\mathfrak{l}}$.

Таким образом, $L^{(2)}$-геометрия $(\bar{\mathfrak{T}}, \bar{\mathfrak{L}})$ не является $TCL^{(2)}$-геометрией.

Исходя из $L^{(2)}$-геометрии $(\bar{\mathfrak{T}}, \bar{\mathfrak{L}})$, постоим объект $(\bar{\mathfrak{T}}, \bar{\bar{\mathfrak{L}}})$ следующим образом:

$$\bar{\bar{\mathfrak{L}}} \overset{\text{деф}}{=\!=} \bar{\mathfrak{L}} \cup \Delta,$$

где

$$\bar{\bar{\mathfrak{l}}} \in \Delta$$

тогда и только тогда когда

$$\bar{\bar{\mathfrak{l}}} = \{x,y,z,u\},\ \{x,y,z,u\} \subseteq \mathfrak{T},^{3)}\ |\{x,y,z,u\}| = 4$$

и

а) существуют $\mathfrak{l}_1, \mathfrak{l}_1' \in L_1$,[3] $\mathfrak{l}_1 \neq \mathfrak{l}_1'$ и $\mathfrak{l}_2, \mathfrak{l}_2 \in L_2$,[3]

---

[1] [8 - 9].

[2] Рис. 2_3.

[3] $(\mathfrak{T}, \{L_1, L_2, L_3, L_4\})$ является 4-сетью изображеной на рис. 2_1.

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$\ell_2 \neq \ell_2'$, таких, что

$$|\bar{\ell} \cap \ell_1| = |\ell \cap \ell_2'| = |\ell \cap \ell_2| = |\ell \cap \ell_2'| = 2^{1)};$$

или

б) существуют в точности одна $\ell_1 \in L_1$ и в точности одна $\ell_2 \in L_2$ такие, что

$$|\bar{\ell} \cap \ell_1| = |\bar{\ell} \cap \ell_2| = 2^{2)}.$$

Отсюда, учитывая 1), непосредственно находим, что $(\bar{\mathfrak{C}}, \bar{\mathfrak{L}})$ является регулярной TCL$^{(2)}$-геометрией. Наконец, учитывая факт, что A_4^4-3-квазигруппы существуют (табл. 1_1-1_4), на основании утверждения 8, находим, что имеет место: A_{10}^4-3-квазигруппы существуют.

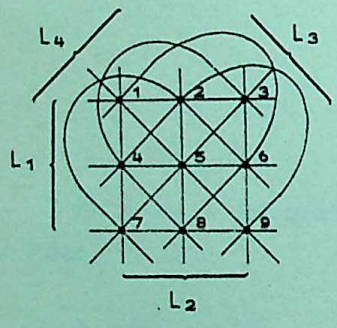

Рис. 2_1

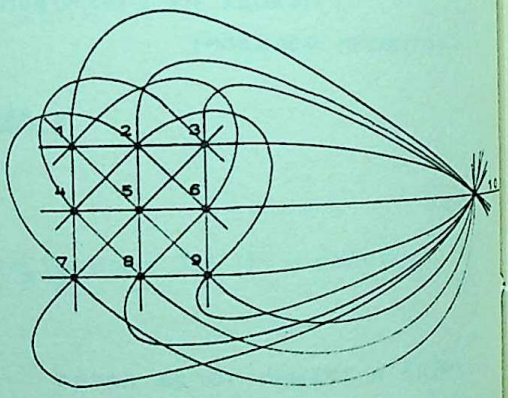

Рис. 2_2

1) Например, как на рис. 2_4.

2) Например, как на рис. 2_5.

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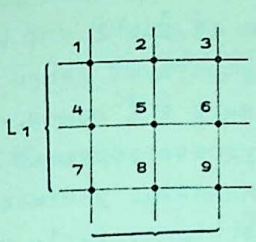

Рис. 2_3

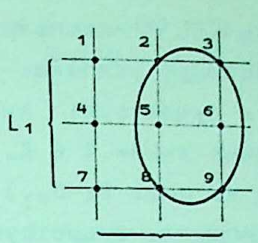

Рис. 2_4 [1)]

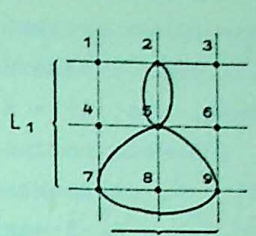

Рис. 2_5 [2)]

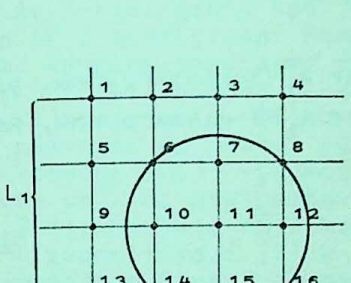

Рис. 3_1

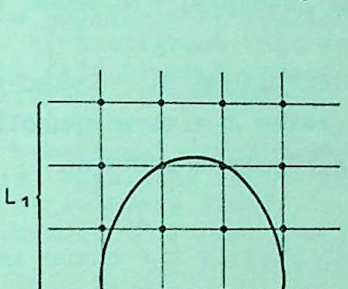

Рис. 3_2

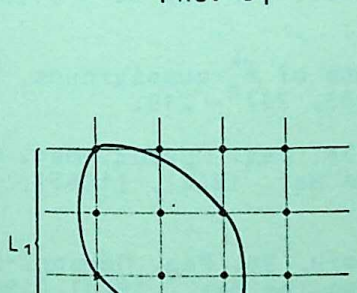

Рис. 3_3

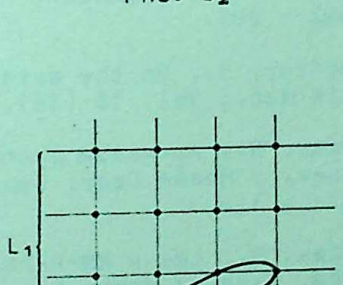

Рис. 3_4

1) $\bar{\bar{\ell}} = \{2,3,8,9\}$.

2) $\bar{\bar{\ell}} = \{2,5,7,9\}$.

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Исходя из 5-сети $(\mathfrak{T}, \{L_1^5\})$ порядка 4, подобными рассуждениями можно построить $TCL^{(2)}$-геометрию $(\bar{\mathfrak{T}}, \bar{\mathfrak{L}})$, $\bar{\mathfrak{T}} = \mathfrak{T} \cup \{17\}$, $\mathfrak{T} = \{1,\ldots,16\}$, удовлетворяющую условию: существуют линии $\ell \in \bar{\mathfrak{L}}$ такие, что $|\ell| = 4$ и существуют линии $\ell \in \bar{\mathfrak{L}}$ такие, что $|\ell| = 5$. (Примеры построения линии $\ell \in \mathfrak{L}$, удовлетворяющие условию $|\ell| = 4$, изображены на рис. 3_1-3_4.) Наконец, учитывая факт, что A_4^4- и A_5^5-3-квазигруппы существуют (табл. 1_1-1_4 и 1_1-1_5), на основании утверждения 8, находим, что имеет место: существует A_{17}-3-квазигруппа, не являющаяся A_{17}^m-3-квазигруппой.

Мы доказали, что имеет место:

УТВЕРЖДЕНИЕ 9. Существуют A_t^m-3-квазигруппы, удовлетворяющие условию $t \neq m$, и существуют A_t-3-квазигруппы, не являющиеся A_t^m-3-квазигруппами.

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REZIME

A_t^m- i A_t-3-kvazigrupe

U radu se uvode A_t^m-3-kvazigrupe kao jedno uopstenje A_t^m-kvazigrupa [2], odnosno A_t^3- i A_t^4-algebri [1]. U radu se uvode i A_t-3-kvazigrupe kao jedno uopstenje A_t-kvazigrupa [4], odnosno A_t^m-3-kvazigrupa. Pomoću A_t^m-kvazigrupa mogu se koordinatizirati konacne regularne ravni čije prave ℓ zadovoljavaju uslov $|\ell| \geq 3$ [3], a pomoću A_t-kvazigrupa TCL-geometrije cije prave ℓ zadovoljavaju uslov $|\ell| \geq 3$ [4]. U ovom radu je, izmedju ostalog, pokazano da svakoj A_t-3-kvazigrupi (A_t^m-3-kvazigrupi) odgovara konacna $TCL^{(2)}$-geometrija (regularna $TCL^{(2)}$-geometrija). Obrnuto važi uz pretpostavku da za svaku liniju ℓ, $|\ell| = m \in N \setminus \{1,2,3\}$, postoji A_m^m-3-kvazigrupa. (Pri tom, ako je $(\mathcal{T},\mathcal{L})$ $TCL^{(2)}$-geometrija i $|\ell| \geq 3$ za svako $\ell \in \mathcal{L}$, onda je $\mathcal{L}$ partitija Harmantisa tipa 3 skupa $\mathcal{T}$ [6 - 7].)

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LN-к- И RN-к-ПОЛУСЕТИ

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РЕЗЮМЕ

к-Полусети, описанные автором в [1], являются одним из обобщений к-сетей [2-3]. к-Полусети весьма тесно связаны со специальными ортогональными системами частичных квазигрупп [1], со специальными кодами [12-16], с аффинными пространствами Спернера [6-7] и с r-дизайнами [17]. В [4] получена характеризация к-полусетей $(\mathfrak{T},\{L_1,\ldots,L_k\})$ с помощью объектов типа $(\mathfrak{T},\mathfrak{L},\|)$, где $\mathfrak{T}\neq\emptyset$, $\mathfrak{L}\subseteq P(\mathfrak{T})\setminus\{\emptyset\}$, $\|\subseteq\mathfrak{L}^2$, $A\|$ удовлетворяет "условию евклидовой параллельности". (Подобная проблематика рассматривается и в [5].). В настоящей работе обобщаются к-полусети на LN-к-полусети и RN-к-полусети (левые-почти-к-полусети и правые-почти-к-полусети). Справедливо: $(\mathfrak{T},\{L_1,\ldots,L_k\})$ является к-полусетью тогда и только тогда, когда $(\mathfrak{T},\{L_1,\ldots,L_k\})$ является сразу LN- и RN-к-полусетью. Существуют LN-к-полусети $(\mathfrak{T},\{L_1,\ldots,L_k\})$, имеющие носитель $(\mathfrak{T},\mathfrak{L})$,

$$\mathfrak{L}=\bigcup_{i=1}^{k}L_i,$$

недопустимый для к-полусетей. Так же существуют RN-к-полусети, имеющие носитель $(\mathfrak{T},\mathfrak{L})$,

$$\mathfrak{L}=\bigcup_{i=1}^{k}L,$$

недопустимый для LN-к-полусетей. Носитель каждой LN-k-полусети является L-геометрией (примечание 8) [8]. Однако, существуют RN-к-полусети, носитель, которых не является L-геометрией. Любую

AMS Mathematics Subject Classification (1980): 20N05.

Key words and phrases: LN-k-seminets, RN-k-seminets, k-seminets, L-geometry, A_t-, A_t^m-quasigroups.

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конечную LN-к-полусеть, в которой каждая прямая имеет по меньшей мере три точки и любые две различные точки являются коллинеарными, можно координатизировать с помощью одной A_t-квазигруппы [8]. Получается характеризация LN-к-полусетей $(\mathfrak{T},\{L_1,\ldots,L_к\})$ с помощью объектов типа $(\mathfrak{T},\mathfrak{L},\|)$, где $\mathfrak{T} \neq \emptyset$, $\mathfrak{L} \subseteq P(\mathfrak{T}) \setminus \{\emptyset\}$, $\| \subseteq \mathfrak{L}^2$, $A\|$ удовлетворяет условию: для каждой $A \in \mathfrak{T}$ и каждой $p \in \mathfrak{L}$ существует не больше, чем одна $p' \in \mathfrak{L}$ такая, что $p' \| p$ и $A \in p'$.

* ☆ *

ОПРЕДЕЛЕНИЕ 1. Пусть $\mathfrak{T}$ непустое множество, элементы которого назовем точки и пусть $\mathfrak{L}$ непустое множество некоторых непустых подмножеств множества $\mathfrak{T}$, элементы которого назовем линии (или прямые). Пусть множества $L_1,\ldots,L_к$, $к \in N \setminus\{1,2\}$, разбивают множество $\mathfrak{L}$. Объект $(\mathfrak{T},\{L_1,\ldots,L_к\})$ назовем LN-к-полусетью тогда и только тогда, когда выполняются следующие условия:

M1 Пересечение каждых двух прямых, принадлежаших различным классам L_i, L_j, $i,j \in \{1,\ldots,к\}$, является одноэлементным множеством или пустым множеством[1];

$LM2_1$ Каждая точка из $\mathfrak{T}$ находится не больше чем в одной прямой каждого класса L_i, $i \in \{1,\ldots,к\}$, и

$M2_2$ Каждая точка из $\mathfrak{T}$ находится по меньше мере в одной прямой.

Объект $(\mathfrak{T},\mathfrak{L})$, где $\mathfrak{L} = \bigcup_{i=1}^{к} L_i$, назовем носителем LN-к-полусети $(\mathfrak{T},\{L_1,\ldots,L_к\})$.

ПРИМЕЧАНИЕ 1. Если в определении 1_L вместо $LM2_1$ и $M2_2$ берется

M2 Каждая точка из $\mathfrak{T}$ находится в одной и только в одной прямой каждого класса L_j, $i \in \{1,\ldots,к\}$;

1) Удобнее: две прямые различных классов пересекаются не больше, чем в одно точке.

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то $(\mathfrak{C},\{L_1,\ldots,L_k\})$ станет к-полусетью [1][1]. Если, кроме того, вместо M1 берется

M1′ Пересечение каждых двух прямых, принадлежащих различным классам L_i, L_j, $i,j \in \{1,\ldots,к\}$, является одноэлементным множеством[2];

то $(\mathfrak{C}, \{L_1,\ldots,L_к\})$ станет к-сетью [2 - 3]. Непосредственно находим, что имеют место импликации:

$$M2 \Rightarrow LM2_1 \wedge M2_2 \quad и \quad M1' \Rightarrow M1.$$

Отсюда получаем, что каждая к-сеть является к-полусетью и каждая к-полусеть является LN-к-полусетью.

УТВЕРЖДЕНИЕ 1. В LN-к-полусетях $(\mathfrak{C},\{L_1,\ldots,L_к\})$ прямые, принадлежащие одному и тому же классу L_i, $i \in \{1,\ldots,к\}$, не пересекаются.

ДОКАЗАТЕЛЬСТВО. Утверждение является следствием аксиомы $LM2_1$. Именно предположение, что существуют различные прямые, принадлежащие одному и тому же классу, имеющиеся непустое пересечение, противречит условию $LM2_1$.

Учитывая определение 1_L, примечание 1 и факт, что в к-полусетях $(\mathfrak{C},\{L_1,\ldots,L_к\})$ справедливо $|L_i| \geq 2$ для каждого $i \in \{1,\ldots,к\}$, находим, что имеет место следующее утверждение:

УТВЕРЖДЕНИЕ 2. Пусть $(\mathfrak{C},\{L_1,\ldots,L_к\})$ к-полусеть. Пусть, далее, $\bar{L}_i$ и $\bar{L}_{i+1}$ непустые множества такие, что $\bar{L}_i \cup \bar{L}_{i+1} = L_i$ и $\bar{L}_i \cap \bar{L}_{i+1} = \emptyset$. Тогда $(\mathfrak{C},\{L^{i-1},\bar{L}_i,\bar{L}_{i+1},L^k_{i+1}\})$ является LN-(к+1)-полусетью, не удовлетворяющей условию M2 (т.е. не

---

1) к-Полусети введены в [1]. Там же получено, что к-полусети можно координатизировать с помощью одного класса ортогональных систем частичных квазигрупп.

2) Удобнее: две прямые различных классов пересекаются в одной и только в одной точке.

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являющейся (к+1)-полусетью).

ПРИМЕЧАНИЕ 2.

2_1. Для каждой LN-t-полусети, полученной способом из утверждения 2 будем сказать, что она является декомпозитом к-полусети.

2_2. В частности, объект $(\mathfrak{T},\{\{\ell_1\},\ldots,\{\ell_t\}\})$, где $\mathfrak{L} = \{\ell_i \mid i \in \{1,\ldots,t\}\}$, $|\mathfrak{L}| = t \in N \setminus \{1,2\}$, и где имеет место импликация $i \neq j \Rightarrow |\ell_i \cap \ell_j| \leq 1$ для любых $i,j \in \{1,\ldots,t\}$ и справедливо $M2_2$, является LN-t-полусетью. Удобнее ее назвать тривиальной. Таким образом, каждую конечную проективную плоскость $(\mathfrak{T},\mathfrak{L})$, $|\mathfrak{L}| = t$, можно считать носителем одной тривиальной LN-t-полусети.

Пусть $\mathfrak{T} = \{1,2,3,4,5,6\}$, и пусть $\mathfrak{L} = \{p_i \mid i \in \{1,\ldots,15\}\}$, где множества p_i, $i \in \{1,\ldots,15\}$, определены следующим образом:

$p_1 = \{1,2\}$, $p_2 = \{3,6\}$,

$p_3 = \{4,5\}$, $p_4 = \{1,6\}$,

$p_5 = \{2,5\}$, $p_6 = \{3,4\}$,

$p_7 = \{2,3\}$, $p_8 = \{1,4\}$,

$p_9 = \{5,6\}$, $p_{10} = \{1,5\}$,

$p_{11} = \{2,4\}$, $p_{12} = \{3,5\}$,

$p_{13} = \{2,6\}$, $p_{14} = \{1,3\}$,

$p_{15} = \{4,6\}$;

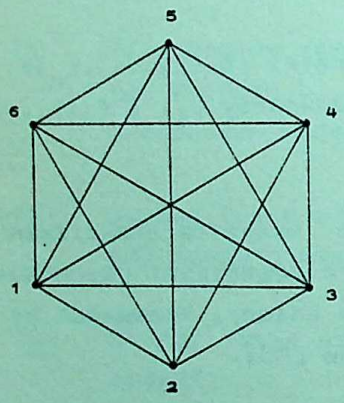

Рис. 1.

см. и рис. 1.

В [5] доказано, что для каждого разбиения $\{L_1,\ldots,L_k\}$ только что описанного множества, объект $(\mathfrak{T},\{L_1,\ldots,L_k\})$ не является к-полусетью. Однако, множество $\{L_1,\ldots,L_6\}$, где

$L_1 = \{p_1, p_2, p_3\}$, $L_2 = \{p_4, p_5, p_6\}$,

$L_3 = \{p_7, p_8, p_9\}$, $L_4 = \{p_{10}, p_{11}\}$,

$L_5 = \{p_{12}, p_{13}\}$, $L_6 = \{p_{14}, p_{15}\}$,

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является таким разбиением множества $\mathcal{L}$, что объект $(\mathcal{T},\{L_1,\ldots,L_6\})$ является LN-6-полусетью. Отсюда получаем, что имеет место следующее утверждение:

УТВЕРЖДЕНИЕ 3. Существуют множества $\mathcal{T}$ и $\mathcal{L} \subseteq P(\mathcal{T}) \setminus \{\emptyset\}$ такие, что имеет место:

а) существует разбиение $\{L_1,\ldots,L_k\}$ множества $\mathcal{L}$ такое, что объект $(\mathcal{T},\{L_1,\ldots,L_k\})$ является нетривиальной LN-k-полусети; и (сразу)

б) для каждого разбиения $\{L_1,\ldots,L_k\}$ множества объект $(\mathcal{T},\{L_1,\ldots,L_k\})$ не является k-полусетью.

ОПРЕДЕЛЕНИЕ $1'_t$. Носитель $(\mathcal{T},\mathcal{L})$ LN-k-полусети $(\mathcal{T},\{L_1,\ldots,L_k\})$ назовем s-допустимым тогда и только тогда, когда существует разбиение $\{\bar{L}_1,\ldots,\bar{L}_k\}$ множества $\mathcal{L}$ $(=\bigcup_{i=1}^{k} L_i)$ такое, что объект $(\mathcal{T},\{\bar{L}_1,\ldots,\bar{L}_k\})$ является t-полусетью. В противоположном случае скажем, что LN-k-полусеть $(\mathcal{T},\{L_1,\ldots,L_k\})$ имеет s-недопустимый носитель.

ПРИМЕЧАНИЕ 3. Описанная LN-6-полусеть имеет s-недопустимый носитель. Конечная проективная плоскость $(\mathcal{T},\mathcal{L})$, $|\mathcal{L}| = t$, является s-недопустимым носителем тривиальной LN-t-полусети $(\mathcal{T},\{\{\ell\}\mid \ell \in \mathcal{L}\})$. Однако, каждый декомпозит k-полусети, ясно, имеет s-допустимый носитель.

ОПРЕДЕЛЕНИЕ $1''_t$. LN-k-полусеть $(\mathcal{T},\{L_1,\ldots,L_k\})$ назовем ℓ-LN-k-полусетью, $\ell \in N$, тогда и только тогда, когда имеет место:

$\underline{M2'_2}$ каждая точка из $\mathcal{T}$ находится в ℓ и только в ℓ прямых.

Описанная LN-6-полусеть является 5-LN-6-полусетью.

Учитывая определение k-полусети (примечание 1) как и определения $1_t, 1'_t, 1''_t$, находим, что имеет место следующее утверждение

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УТВЕРЖДЕНИЕ 4. Если LN-к-полусеть не является ℓ-LN-к-полусетью, то она имеет s-недопустимый носитель.

УТВЕРЖДЕНИЕ 5. Если объект $(\mathfrak{T},\{L_1,\ldots,L_к\})$ является ℓ-LN-к-полусетью, тогда $\ell \leq к$.

ДОКАЗАТЕЛЬСТВО. Утверждение является следствием аксиомы $LM2_1$ и $M2_2'$. Именно, так как через каждую точку проходит в точности ℓ прямых $(M2_2')$, предположение, что $\ell > к$ влечет в контрадикцию с $LM2_1$.

Подобными рассуждениями доказывается и следующее утверждение:

УТВЕРЖДЕНИЕ 5'. Если объект $(\mathfrak{T},\{L_1,\ldots,L_к\})$ является LN-к-полусетью и $\ell(A)$ является числом прямых, проходящих через точку A, тогда $\ell(A) \leq к$ для каждой точки $A \in \mathfrak{T}$.

ТЕОРЕМА 6. ℓ-LN-к-полусеть $(\mathfrak{T},\{L_1,\ldots,L_к\})$ является к-полусетью тогда и только тогда, когда $\ell = к$.

ДОКАЗАТЕЛЬСТВО.

а) $\Rightarrow$

Утверждение в этом направлении является непосредственным следствием определения к-полусети (примечание 1) и определения 1_L.

б) $\Leftarrow$

Если $\ell = к$, то, на основании $M2_2'$, через каждую точку проходит в точности к прямых. Отсюда, ввиду $LM2_1$, заключаем, что в $(\mathfrak{T},\{L_1,\ldots,L_к\})$ имеет место M2. Именно, так как через каждую точку проходит в точности к прямых, то из предположения, что существует точка $A \in \mathfrak{T}$ не принадлежащая ни одной прямой некоторого класса L_i, заключаем, что существуют прямые $\ell_j \neq \ell_{j'}$ принадлежащие одному и тому же классу $L_j \neq L_i$ такие, что $\ell_j \cap \ell_{j'} = \{A\}$. Так как это влечет в контрадикцию с $LM2_1$, заключаем, что утверждение доказано и в направлении $\Leftarrow$.

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Учитывая теорему 6 и примечание 1, находим, что имеет место и следующее утверждение:

СЛЕДСТВИЕ 7. ℓ-LN-к-полусеть $(\mathfrak{T},\{L_1,\ldots,L_к\})$ является к-сетью тогда и только тогда, когда $\ell = к$ и имеет место M1′.

ТЕОРЕМА 8. Если в ℓ-LN-к-полусети $(\mathfrak{T},\{L_1,\ldots,L_к\})$ справедливы M1′ и $\ell < к$, тогда $(\mathfrak{T},\{L_1,\ldots,L_к\})$ имеет s-недопустимый носитель.

ДОКАЗАТЕЛЬСТВО. Так как $\ell < к$, в виду теорем 6, заключаем, что в $(\mathfrak{T}_к,\{L_1,\ldots,L_к\})$ не имеет место M2. Для данных множеств $\mathfrak{T}$ и $\mathfrak{L} = \bigcup_{i=1}^{\ell} L_i$, $\ell \in N$ является фиксированным. Далее, на основании M1′ и утверждения 1, находим, что максимальные множества попарно не пересекающихся прямых являются, в точности, множества $L_1,\ldots,L_к$. Отсюда находим, что каждое разбиение множества $\mathfrak{L}$, имеющее число классов меньше, чем к, имеет по меньшей мере один класс, в котором существуют пересекающиеся прямые. Утверждение доказано.

На рис. 3 - 5 изображены, в том же порядке, 2-LN-3--полусеть, 2-LN-4-полусеть и 3-LN-5-полусеть, удовлетворяющие условию M1′. Таким образом, имеет место следующее утверждение:

УТВЕРЖДЕНИЕ 9. Существуют ℓ-LN-к-полусети, в которых имеет место M1′ и имеют s-недопустимый носитель.

На рис. 2 изображена LN-3-полусеть, в которой имеет место M1′ и не являющаяся ℓ-LN-3-полусетью. Отсюда, учитывая утверждение 4, находим, что имеет место следующее утверждение:

УТВЕРЖДЕНИЕ 10. Существуют LN-к-полусети, не являющеся ℓ-LN-к-полусетями, в которых имеет место M1′ и имеют s-недопустимый носитель.

УТВЕРЖДЕНИЕ 11_1. Пусть $(\mathfrak{T},\{L_1,\ldots,L_к\})$ LN-к-полусеть,

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a $\|$ бинарное отношение в $\mathfrak{L} = L_1 \cup \ldots \cup L_k$, определяемое следующим образом:

(в) $$p \| q \overset{\text{деф}}{\Leftrightarrow} p \in L_i \wedge q \in L_j \wedge i = j.$$

Тогда в объекте $(\mathfrak{T}, \mathfrak{L}, \|)$ имеет место:

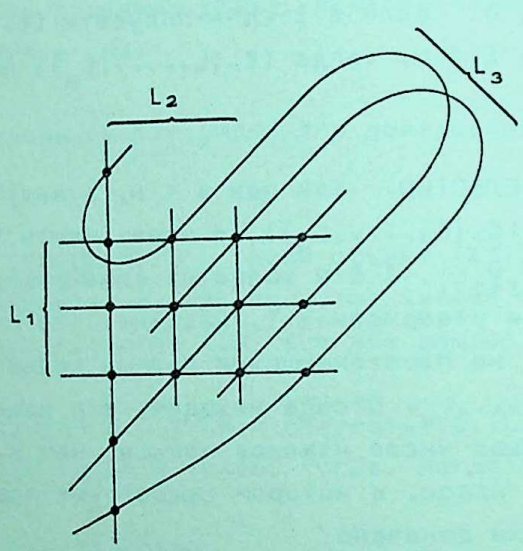

Рис. 2.

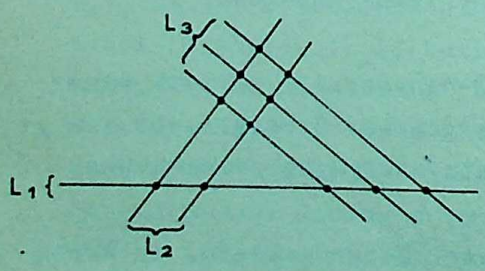

Рис. 3.

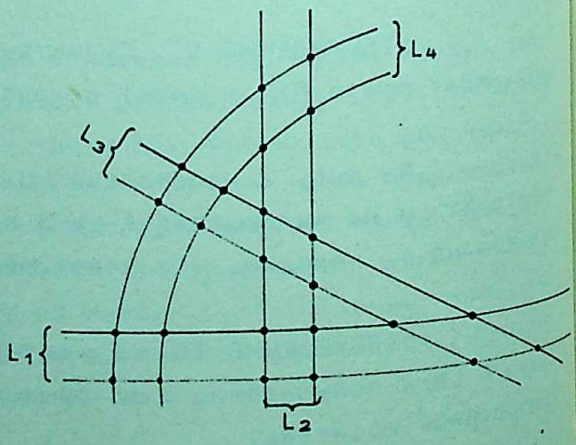

Рис. 4.

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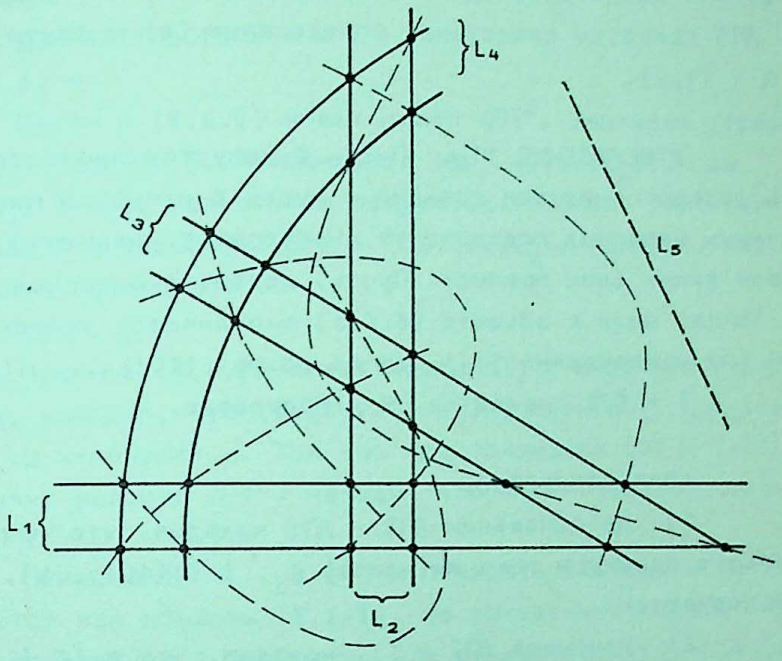

Рис. 5.

ЛП1. каждая точка $A \in \mathfrak{T}$ находится в $\ell(A) \in N$ прямых;

ЛП2 через любые две различные точки проходит не более одной прямой;

ЛП3 $\parallel$ является РСТ отношением в $\mathfrak{L}$;

ЛП4 для каждой $A \in \mathfrak{T}$ и каждой $p \in \mathfrak{L}$ существует не более чем одна $p' \in \mathfrak{L}$ такая, что имеет место

$$A \in p' \wedge p' \parallel p;$$

и

ЛП5 $$|\mathfrak{L}/\parallel| = \kappa \in N \setminus \{1,2\}.$$

ДОКАЗАТЕЛЬСТВО. ЛП1 является следствием аксиомы $M2_2$,

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утверждения 5' и факта, что $k \in N$. ЛП2 является следствием аксиомы М1 и утверждения 1. ЛП3 является следствием определения (e) и факта, что $\{L_1,\ldots,L_k\}$ является разбиением множества $\mathfrak{L}$. ЛП4 является следствием аксиомы $LM2_1$ и определения (e). ЛП5 является следствием определения (e) и факта, что $k \in N \setminus \{1,2\}$.

УТВЕРЖДЕНИЕ 11_2. Пусть $\mathfrak{T}$ непустое множество, элементы которого назовем точками и пусть $\mathfrak{L}$ непустое множество некоторых непустых подмножеств множества $\mathfrak{T}$, элементы которого назовем линии (или прямые). Пусть, далее, $\|$ бинарное отношение в $\mathfrak{L}$. Тогда, если в объекте $(\mathfrak{T},\mathfrak{L},\|)$ выполняются условия ЛП1 - - ЛП5 (из утверждения 11_1), тогда объект $(\mathfrak{T},\{L_1,\ldots,L_k\})$, где $\{L_1,\ldots,L_k\} = \mathfrak{L}/\|$, является LN-k-полусетью.

ДОКАЗАТЕЛЬСТВО.

1) На основании ЛП3 - ЛП5 находим, что прямые, принадлежащие одному и тому же классу L_i, $i \in \{1,\ldots,k\}$, попарно не пересекаются;

2) Учитывая ЛП2 и 1), находим, что в $(\mathfrak{T},\{L_1,\ldots,L_k\})$ имеет место М1;

3) Учитывая 1) и ЛП4, находим, что в $(\mathfrak{T},\{L_1,\ldots,L_k\})$ имеет место $LM2_1$; и

4) Учитывая ЛП1, находим, что в $(\mathfrak{T},\{L_1,\ldots,L_k\})$ имеет место $M2_2$.

ПРИМЕЧАНИЕ 4. Ввиду утверждений 11_1 и 11_2, имеет смысл объект $(\mathfrak{T},\mathfrak{L},\|)$, удовлетворяющий условиям ЛП1 - ЛП5, считать LN-k-полусетью. Отношение $\|$ назовем отношением параллельности (или: параллельностью).

ТЕОРЕМА 12.. Если объект $(\mathfrak{T},\mathfrak{L},\|)$ является LN-k-полусетью, $k = |\mathfrak{L}/\||$, $k \in N \setminus \{1,2\}$ [1], тогда эквивалентны следующие высказания:

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[1] См. Примечание 4.

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ЛП1′ $(\forall A \in \mathfrak{T})\ell(A) = \kappa$; и

ЛП4′ $(\forall A \in \mathfrak{T})(\forall p \in \mathfrak{L})(\exists! p' \in \mathfrak{L})(p' \parallel p \wedge A \in p')$.

ДОКАЗАТЕЛЬСТВО.

а) $\Rightarrow$

Пусть в $(\mathfrak{T},\mathfrak{L},\parallel)$ имеет место ЛП1′. Учитывая утверждение 11_2 и теорему 6, находим,что $(\mathfrak{T},\{L_1,\ldots,L_\kappa\})$, где $\{L_1,\ldots,L_\kappa\} = \mathfrak{L}/\parallel$, является к-полусетью. Отсюда, учитывая М2, утверждение 1 и равенство $\mathfrak{L}/\parallel = \{L_1,\ldots,L_\kappa\}$, находим, что в $(\mathfrak{T},\mathfrak{L},\parallel)$ имеет место ЛП4′.

б) $\Leftarrow$

Пусть в $(\mathfrak{T},\mathfrak{L},\parallel)$ имеет место ЛП4′. Учитывая утверждение 11_2, находим, что $(\mathfrak{T},\{L_1,\ldots,L_\kappa\})$, где $\{L_1,\ldots,L_\kappa\} = \mathfrak{L}/\parallel$, является LN-к-полусетью. Так как предположение $(\exists A \in \mathfrak{T})\ell(A) < \kappa$ противоречит условию ЛП4′, находим что имеет место условие ЛП1′.

ПРИМЕЧАНИЕ 5. В [4] к-полусети $(\mathfrak{T},\{L_1,\ldots,L_\kappa\})$ рассматриваются как объекты $(\mathfrak{T},\mathfrak{L},\parallel)$. На основании теоремы 6 и 12, учитывая примечание 4, находим, что объект $(\mathfrak{T},\mathfrak{L},\parallel)$, где $\mathfrak{T}$, $\mathfrak{L}$ и $\parallel$ имеют значение из утверждения 11_2, имеет смысл считать к-полусетью тогда и только тогда, когда выполняются условия:

1° ЛП1′, ЛП2 - ЛП5;

или

2° ЛП1 - ЛП3, ЛП4′ и ЛП5.

*****

ОПРЕДЕЛЕНИЕ 1_R. Пусть $\mathfrak{T}$ непустое множество, элементы которого назовем точками и пусть $\mathfrak{L}$ непустое множество некоторых непустых подмножеств множества $\mathfrak{T}$, элементы которого назовем линиями. Пусть множества $L_1,\ldots,L_\kappa$, $\kappa \in N \setminus \{1,2\}$, разбывают множество $\mathfrak{L}$. Объект $(\mathfrak{T},\{L_1,\ldots,L_\kappa\})$ назовем RN-к-полусетью тогда и только тогда, когда выполняются следующие условия:

М1 Пересечение каждых двух линий, принадлежащих различ-

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ным классам L_i, L_j, $i,j \in \{1,\dots,k\}$, является одноэлементным множеством или пустым множеством[1];

$RM2_1$ Каждая точка из $\mathfrak{T}$ находится по меньшей мере в одной линии каждого класса L_i, $i \in \{1,\dots,k\}$[2].

Объект $(\mathfrak{T},\mathfrak{L})$, где $\mathfrak{L} = \bigcup_{i=1}^{k} L_i$, назовем <u>носителем</u> RN-k-полусети $(\mathfrak{T},\{L_1,\dots,L_k\})$.

Непосредственным следствием аксиомы $RM2_1$ является:

$M2_2$ Каждая точка из $\mathfrak{T}$ находится по меньшей мере в одной линии.

Учитывая определение 1_L, определение 1_R и определение k-полусети[3], находим, что имеет место следующее утверждение:

УТВЕРЖДЕНИЕ 13. Пусть множества $\mathfrak{T}, L_1, \dots, L_k$ имеют значения из определений 1_L и 1_R. Тогда объект $(\mathfrak{T},\{L_1,\dots,L_k\})$ является k-полусетью если и только, если $(\mathfrak{T},\{L_1,\dots,L_k\})$ является LN- и RN-k-полусетью.

УТВЕРЖДЕНИЕ 14. RN-k-полусеть $(\mathfrak{T},\{L_1,\dots,L_k\})$ является k-полусетью тогда и только тогда, когда имеет место

(o) $$(\forall i \in \{1,\dots,k\})(\forall j \in \{1,\dots,k\})(\forall p \in L_i)(\forall p' \in L_j)$$
$$(p \neq p' \wedge i = j \Rightarrow p \cap p' = \emptyset)^{4}.$$

ДОКАЗАТЕЛЬСТВО.

а) $\Rightarrow$

Так как каждая k-полусеть является LN-k-полусетью, то, на основании утверждения 1, имеет место (o).

---

1) См. определение 1_L.

2) $(\forall A \in \mathfrak{T})(\exists p_1 \in L_1)\dots(\exists p_k \in L_k)(A \in p_1 \wedge \dots \wedge A \in p_k)$

3) См. Примечание 1.

4) Пересечение каждых двух различных линий, принадлежащих одному и тому же классу является пустым множеством.

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б) ⇐

Пусть в RN-к-полусети $(\mathfrak{T},\{L_1,\ldots,L_к\})$ имеет место (о). Через каждую точку проходит по меньшей мере одна линия из каждого класса L_i, $i \in \{1,\ldots,к\}$ ($RM2_1$). Отсюда, учитывая (о), находим, что в $(\mathfrak{T},\{L_1,\ldots,L_к\})$ имеет место M2, т.е. что $(\mathfrak{T},\{L_1,\ldots,L_к\})$ является к-полусетью.

Учитывая M2 (примечание 1) и определение 1_R, находим, что имеет место и следующее утверждение:

УТВЕРЖДЕНИЕ 15. Пусть $(\mathfrak{T},\{L_1,\ldots,L_к\})$ к-полусеть и $к \geq 4$. Тогда $(\mathfrak{T},\{L_1^{i-1},L_i \cup L_{i+1},L_{i+2}^{k}\})$ является RN-(к-1)-полусетью не удовлетворяющей условию M2 (т.е. не являющейся (к-1)-полусетью).[1)]

ПРИМЕЧАНИЕ 7. Для каждой RN-t-полусети, полученной способом из утверждения 15, скажем, что она является композитом к-полусети. 5-LN-6-полусеть, носитель которой изображен на рис. 1 имеет s-недопустимый носитель. В этой 5-LN-6-полусети имеет место: $|L_1| = |L_2| = |L_3| = 3$, $|L_4| = |L_5| = |L_6| = 2$, через каждую точку проходит одна и только одна линия из каждого класса L_i, $i \in \{1,2,3\}$, и через каждую точку проходят а точности две линии, принадлежащие множеству $\bar{L}_4 = L_4 \cup L_5 \cup L_6$. Так как $\{L_1,L_2,L_3,\bar{L}_4\}$ также является разбиением множества $\mathfrak{L}$, отсюда получаем, что $(\mathfrak{T},\{L_1,L_2,L_3,\bar{L}_4\})$ является RN-4-полусетью, не удовлетворяющей условию $LM2_1$. (В таких случаях позволим себе сказать, что речь идет о композите LN-к-полусети со s-недопустимым носителем.)

На рис. 6 изображено одно из разбиений множества $\mathfrak{L}$, где $|\mathfrak{L}| = 16$ и $|\mathfrak{T}| = 16$. Объект $(\mathfrak{T},\{L_1,L_2,L_3,L_4\})$ не является 4-сетью, так как существуют линии $p \in L_3$ и $q \in L_4$, пересекающиеся в двух различных точках. Заметим, что объекты $(\mathfrak{T},\{L_1,L_2,L_3,\})$ и $(\mathfrak{T},\{L_1,L_2,L_4\})$ являются 3-сетями. Так как множество $\{L_1,L_2,L_3 \cup L_4\}$ также является разбиением множества $\mathfrak{L}$, отсюда находим,

[1)] См. утверждение 2.

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что объект $(\mathfrak{T}, \{L_1, L_2, L_3 \cup L_4\})$ является RN-3-полусетью. Далее, учитывая факт, что носитель этой RN-3-полусети не является носителем ни одной LN-к-полусети (ввиду M1 и утверждения 1)[1], находим, что имеет место следующее утверждение:

УТВЕРЖДЕНИЕ 16. Существуют множества $\mathfrak{T}$ и $\mathfrak{L} \subseteq P(\mathfrak{T}) \setminus \{\emptyset\}$ такие, что имеет место:

а) существует разбиение $\{L_1, \ldots, L_к\}$ множества $\mathfrak{L}$ такое, что объект $(\mathfrak{T}, \{L_1, \ldots, L_к\})$ является RN-к-полусетью; и

б) не существует разбиение $\{\bar{L}_1, \ldots, \bar{L}_к\}$ множества $\mathfrak{L}$ такое, что объект $(\mathfrak{T}, \{\bar{L}_1, \ldots, \bar{L}_к\})$ является LN-$\bar{к}$-полусетью.

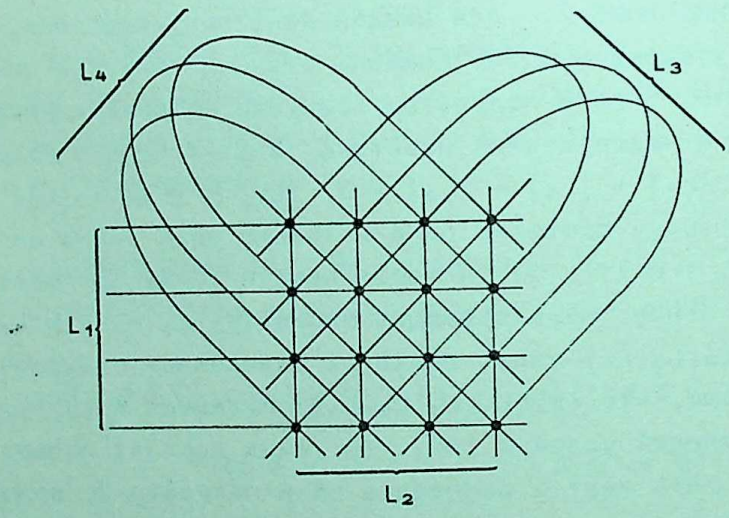

Рис. 6.

ОПРЕДЕЛЕНИЕ $1'_R$. Носитель $(\mathfrak{T}\ \mathfrak{L})$ RN-к-полусети $(\mathfrak{T}, \{L_1, \ldots, L_к\})$ назовем LS-допустимым (s-допустимым) тогда и

[1] Заметим, что кроме этого, имеет место: через каждые три попарно различные точки проходит не более одной линии.

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только тогда, когда существует разбиение $\{\bar{L}_1,\ldots,\bar{L}_t\}$ множества $\mathcal{L}$ ($= \bigcup_{i=1}^{k} L_i$) такое, что объект $(\mathcal{T},\{\bar{L}_1,\ldots,\bar{L}_t\})$ является LN-t-полусетью (t-полусетью). В противоположном случае скажем, что RN-к-полусеть $(\mathcal{T},\{L_1,\ldots,L_к\})$ имеет LS-недопустимый (s-недопустимый) носитель.

На рис. 7 изображено одно из разбиений $\{L_1,L_2,L_3,L_4\}$ множества $\mathcal{L}$, где $|\mathcal{L}| = 15$ и $|\mathcal{T}| = 16$. Объект $(\mathcal{T},\{L_1,L_2,L_3 \cup L_4\})$ является RN-3-полусетью, имеющей LS-недопустимый носитель. Кроме того, здесь $\ell(A) \in \{3,4\}$, где $\ell(A)$ число линий, проходящих через точку $A \in \mathcal{T}$. Заметим, что в RN-3-полусети, изображенной на рис. 6 $\ell(A) = 4$ для всех $A \in \mathcal{T}$.

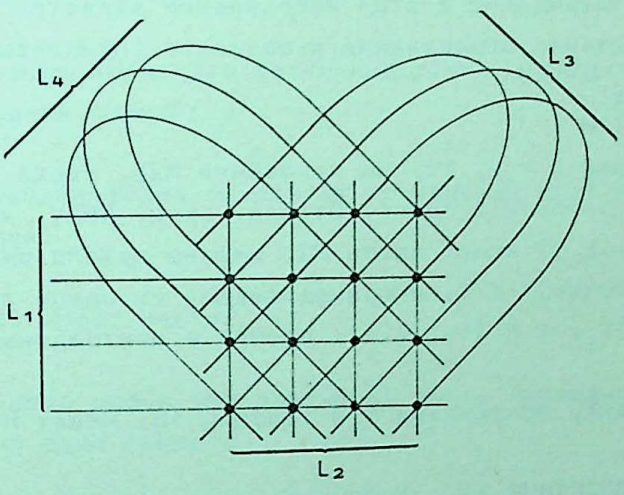

Рис. 7.

ОПРЕДЕЛЕНИЕ $1'_R$. RN-к-полусеть $(\mathcal{T},\{L_1,\ldots,L_к\})$ назовем ℓ-RN-к-полусетью, $\ell \in N$, тогда и только тогда, когда имеет место:

$M2'_2$ каждая точка из $\mathcal{T}$ находится в ℓ и только в ℓ линии.

Следующее утверждение является следствием условий $RM2_1$ и $M2'_2$.

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УТВЕРЖДЕНИЕ 17. Если объект $(\mathfrak{T},\{L_1,\ldots,L_к\})$ является ℓ-RN-к-полусетью, то $\ell \geq к$.

Более того, имеет место:

УТВЕРЖДЕНИЕ 17'. Если объект $(\mathfrak{T},\{L_1,\ldots,L_к\})$ является RN-k-полусетью и $\ell(A)$ числом линий, проходящих через точку точку $A \in \mathfrak{T}$, тогда $\ell(A) \geq к$ для каждой точки $A \in \mathfrak{T}$.

ТЕОРЕМА 18. ℓ-RN-к-полусеть $(\mathfrak{T},\{L_1,\ldots,L_к\})$ является к-полусетью тогда и только тогда, когда $\ell = к$.

ДОКАЗАТЕЛЬСТВО.

а) $\Rightarrow$

Утверждение в этом направлении является непосредственным следствием определения к-полусети (примечание 1) и определения 1_R.

б) $\Leftarrow$

Если $\ell = к$, то, на основании $M2'_2$, через каждую точку проходит в точности к линии. Отсюда, ввиду $RM2_1$, находим, что в $(\mathfrak{T},\{L_1,\ldots,L_к\})$ имеет место M2. Именно предположение, что существует точка $A \in \mathfrak{T}$, непринадлежащая ни одной линии некоторого класса L_i, $i \in \{1,\ldots,к\}$, влечет в контрадикцию с $RM2_1$.

Ввиду примечания 1 и теоремы 18, имеет место:

СЛЕДСТВИЕ 19. ℓ-RN-к-полусеть $(\mathfrak{T},\{L_1,\ldots,L_к\})$ является к-сетью тогда и только тогда, когда выполняется равенство $\ell = к$ и имеет место M1'.

4-RN-3-полусеть, изображенная на рис. 6, имеет LS--недопустимый носитель. Все-таки она удовлетворяет условию M1'. Таким образом, имеет место:

УТВЕРЖДЕНИЕ 20. Существуют ℓ-RN-к-полусети, в которых имеет место M1' и имеется LS-недопустимый носитель.

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RN-3-полусеть, изображенная на рис. 7, имеет LS--недопустимый носитель. Все-таки она удовлетворяет условию M1′. Таким образом, имеет место и следующее утверждение:

УТВЕРЖДЕНИЕ 21. Существуют RN-к-полусети, в которых имеет место M1′ и которые имеют LS-недопустимый носитель.

ПРИМЕЧАНИЕ 8. В [8] "L-геометия" имеет следующий смысл:

Пусть $\mathfrak{T}$ непустое множество[1] и пусть непустое множество $\mathfrak{L}$ множество некоторых непустых подмножеств множества $\mathfrak{T}$. Элементы множества $\mathfrak{T}$ называются точками, элементы множества $\mathfrak{L}$ прямыми. Объект $(\mathfrak{T},\mathfrak{L})$ называется L-геометрий тогда и только тогда, когда выполняются следующие условия:

L1 через любые две различные точки проходит не более одной прямой; и

L2 каждая точка находится по меньшей мере в одной прямой.

Если в $(\mathfrak{T},\mathfrak{L})$ имеет место:

L1′ через любые две различные точки проходит одна и только одна пряма ;

то в $(\mathfrak{T},\mathfrak{L})$ справедливы условия L1 - L2.

L-геометрию $(\mathfrak{T},\mathfrak{L})$, в которой имеет место L1′, автор в [8], позволил себе назвать TCL-геометрией (тотально колинеаризированной L-геометрией)[2].

Носители LN-к-полусетей являются L-геометриями. L--геометрии являются носителями и некоторых RN-к-полусетей. Но

---

[1] В [8] речь идет о конечном множестве.

[2] В самом деле, если $(\mathfrak{T},\mathfrak{L})$ является TCL-геометрией и $|\ell| \geq 2$ для всех $\ell \in \mathfrak{L}$, то $\mathfrak{L}$ является разбиением Хартманиса типа 2 множества $\mathfrak{T}$ [18].

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существуют RN-к-полусети, имеющие носители, не являющиеся L-геометриями. Примеры таких RN-к-полусетей изображены на рис. 6 и 7. Для этих носителей имеет место: L2 и

$L^{(2)}1$ через каждые три попарно различные точки проходит не более одной линии.

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REZIME

LN-k- I RN-k-POLUREŠETKE

k-Polurešetke, koje je autor uveo u [1], predstavljaju jedno uopštenje k-rešetaka [2-3]. k-Polurešetke su u tesnoj vezi sa specijalnim ortogonalnim sistemima parcijalnih kvazigrupa [1], sa specijalnim kodovima [12-16], sa afinim prostorima Spernera [6-7] i sa r-dizajnima [17]. U ovom radu se k-polurešetke uopštavaju na LN-k-polurešetke i RN-k-polurešetke (levе-skoro-k-polurešetke i desne-skoro-k-polurešetke). Pri tom važi: objekat $(\mathfrak{T},\{L_1,\ldots,L_k\})$ jeste k-polurešetka akko je LN-k- i RN-k-polurešetka. Izmedju ostalog, u radu se nalazi karakterizacija LN-k-polurešetaka $(\mathfrak{T},\{L_1,\ldots,L_k\})$, pomoću objekata tipa $(\mathfrak{T},\mathfrak{L},\|)$, gde $\mathfrak{T} \neq \emptyset$, $\mathfrak{L} \subseteq P(\mathfrak{T}) \setminus \{\emptyset\}$, $\| \subseteq \mathfrak{L}^2$, a $\|$ zadovoljava uslov: za svako $A \in \mathfrak{T}$ i svako $p \in \mathfrak{L}$ postoji najviše jedna $p' \in \mathfrak{L}$ takva da je $p' \| p$ i $A \in p'$.

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NATURAL EQUIVALENCE OF GRAPH RELATED BIFUNCTORS

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ABSTRACT

We introduce a method of investigating both topological spaces and graphs through a suitable notion of homotopy for certain classes of functions of graphs into hyperspaces of topological spaces. We show that this approach is naturally equivalent to the Demaria´s method which is based on the notion of homotopy for certain classes of functions of topological spaces into graphs.

1. INTRODUCTION.

There has been a considerable research by D.C.Demaria, M.Burzio, G.M.Gianella, C.Guido, B.Casciaro, F.Cammaroto, L.Carini, P.M.Gandini and O.M.Amici on the theory of regular homotopy of regular functions into graphs. This theory can be vaguely described as follows.

One first introduces a class of regular functions from a topological space X into a graph G as functions $f:X \to G$ for which the preimages of nonadjacent vertices are topologically separated. Regular functions $f, g:X \to G$ are homotopic provided there is a regular function $H:X \times [0,1] \to G$ such that

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Key words and phrases: topological spaces, graphs, homotopy.

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$H(x,0)=f(x)$ and $H(x,1)=g(x)$ for every $x\in X$. The regular homotopy theory is simply the study of the bifunctor [5] r which associates to a topological space X and a graph G the set $r[X:G]$ of all equivalence classes of regular functions of X into G with respect to the equivalence relation of homotopy. Of course the bifunctior r determines two functors: r_G (when G is regarded fixed) and r^X (when X is regarded fixed) on the category of topological spaces and on the category of graphs, respectively.

In this paper, we shall reverse the above procedure and consider multivalued functions of a graph G into a topological space X instead of functions of X into G. The multivalued function $\varphi:G\to X$ is regular provided the images of nonadjacent vertices are topologically separated. Two regular multivalued functions $\psi,\varphi:G\to X$ are homotopic provided there is a regular multivalued function $\Phi:G\to X\times[0,1]$ that connects φ and ψ. Thus, we can define a bifunctor c which associates to a topological space X and a graph G the set $c[G:X]$ of all equivalence classes of regular multivalued functions of G into X with respect to the equivalence relation of homotopy.

Our main result shows that the bifunctors r and c are naturally equivalent. It follows that the two approaches are only formally different. However, while Demaria's regular functions could be regarded as a realization (analogous to the intersection graphs [4]) of a graph in terms of covers of a topological space into pairwise disjoint subsets, in our approach this last condition has been eliminated. This has some obvious advantages. For example, our point of view makes the whole subject a part of the hyperspace theory of topological spaces so that we can use the machinery of hyperspaces, multivalued functions, and their selections and easily apply the method to other structures.

We also show that the similar statements hold for other types of regular functions and multivalued functions (completely regular and 0-regular and 0*-regular for directed

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graphs), and for the case of pairs and more generally of n-tuples.

2. R-FUNCTIONS AND R-HOMOTOPY

In this section we shall recall Demaria's [1] notion of a regular function (or an r-function) of a space into a graph and the associated equivalence relation of homotopy for r-functions.

Let G be a graph. A pair $\{v,w\}$ of elements of G will be called a j-pair provided either $v = w$ or v and w are end-points of an edge in G. The pairs of vertices which are not j-pairs are called d-pairs.

A function $f:X \to G$ of a topological space X into a graph G is an r-function provided $\overline{f^{-1}(v) \cap f^{-1}(w) = f^{-1}(v) \cap f^{-1}(w)} = \emptyset$ for every d-pair $(v,w) \in G \times G$. Two r-functions $f,g:X \to G$ are r-homotopic (in notation $f \overset{r}{\simeq} g$) provided there exists an r-function $F:X \times I \to G$ (called an r-homotopy) such that $F(x,0) = f(x)$ and $F(x,1) = g(x)$ for every $x \in X$. The relation of r-homotopy is an equivalence relation on the set $r(X; G)$ of all r-functions of X into G. Let $[f]$ denote the equivalence class of $f \in r(X;G)$ and let $r[X;G]$ denote the quotient set. Moreover, we can consider the association $(X,G) \to r[X;G]$ as a bifunctor [5, p. 39] $r:\tau^{o} \times G \to S$, where τ^{o} is the opposite to the category τ of topological spaces and continuous functions, G is the category of graphs and j-functions (i.e., finite-to-one functions which map j-pairs into j-pairs), and S is the category of sets and functions. On morphisms r is defined as follows. Let $(f,m) \in Mor_{\tau^{o} \times G}((X,G),(Y,H))$. Then $r(f,m):r[X;G] \to r[Y;H]$ is given by $r(f,m)([g]) = [mogof]$ for $g \in r(X; G)$.

For pairs the above notions are defined as follows. In an analogous way it is possible to consider those concepts also for arbitrary n-tuples.

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Let X be a topological space and let Y be a subspace of X. Let G be a graph and let H be a subgraph of G. Call a function $f:(X,Y)\to(G,H)$ between pairs an r-function provided $f:X\to G$ and $f|_Y:Y\to H$ are both r-functions. Two r-functions $f, g:(X,Y)\to(G,H)$ are r-homotopic provided there is an r-function $F:(X\times I, Y\times I)\to(G,H)$ such that $F(x,0)=f(x)$ and $F(x,1)=g(x)$ for every $x\in X$. The relation of r-homotopy is an equivalence relation on the set $r(X,Y;\ G,H)$ of all r-functions from (X,Y) into (G,H). The equivalence class of an $f\in r(X,Y;\ G,H)$ is denoted by $[f]$ while $r[X,Y;\ G,H]$ denotes the set of all equivalence classes.

Just like in the absolute case, we can consider the association $((X,Y),(G,H))\to r[X,Y;\ G,H]$ as a bifunctor $r_2:\tau_2^o\times G_2\to S$, where τ_2^o is the opposite to the category τ_2 of pairs (X,Y) consisting of a topological space X and its subspace Y and continuous functions of such pairs and G_2 is the category of pairs (G,H) consisting of a graph G and a subgraph H of G and j-functions of such pairs.

When (X,Y) is a pair $(I^n, \dot{I}^n)$ consisting of the n-cube and its boundary $\dot{I}^n$ and (G,H) is a pair (G,v) consisting of the graph G and a vertex $v\in G$, the set $r_n(G,v) = r[I^n,\dot{I}^n);\ G,v]$ becomes a group under the binary operation o defined as follows. For $[f], [g]\in r_n(G,v)$, put $[f]o[g]=[h]$ where $h:(I^n,\dot{I}^n)\to(G,v)$ is given by the formula

$$h(t_1,\ldots,t_n)=\begin{cases} f(3t_1,t_2,\ldots,t_n), & t_1\in[0,1/3] \\ f(1,t_2,\ldots,t_n), & t_1\in[1/3,2/3] \\ g(3t_1-2,t_2,\ldots,t_n), & t_1\in[2/3,1] \end{cases}$$

for every $(t_1,\ldots,t_n)\in I^n$.

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3. C-FUNCTIONS AND C-HOMOTOPY

This section introduces a class of multivalued functions (called c-functions) from a graph into a space and the associated equivalence relation of homotopy for c-functions.

Let a^*X denote the hyperspace of all subsets of a topological space X and let $p_X: X \times I \to X$ be the natural projection.

DEFINITION 3.1. *A function* $\varphi: G \to a^*X$ *of a graph* G *into* a^*X *is a c-function provided*

(i) $X = \cup \{\varphi(v) \mid v \in G\}$,

and

(ii) $\overline{\varphi(v)} \cap \varphi(w) = \varphi(v) \cap \overline{\varphi(w)} = \emptyset$

for every d-pair $(v,w) \in G \times G$.

Two c-functions $\varphi, \psi: G \to a^*X$ are c-homotopic (in notation, $\varphi \overset{c}{\simeq} \psi$) provided there exists a c-function $\Phi: G \to a^*(X \times I)$ (called a c-homotopy) such that $\varphi(v) = p_X(\Phi(v) \cap (X \times \{0\}))$ and $\psi(v) = p_X(\Phi(v) \cap (X \times \{1\}))$ for every $v \in G$.

THEOREM 3.2. *The relation* $\overset{c}{\simeq}$ *is an equivalence relation on the set* $c(G; X)$ *of all c-functions of* G *into* X.

P r o o f. a) (Reflexive) For $\varphi \in c(G;X)$, the function $\Phi: G \to a^*(X \times I)$ defined by $\Phi(v) = \varphi(v) \times I$ for each $v \in G$ is a c-function and $p_X(\Phi(v) \cap (X \times \{0\})) = p_X(\Phi(v) \cap (X \times \{1\})) = \varphi(v)$ for every $v \in G$. Hence, $\varphi \overset{c}{\simeq} \varphi$.

b) (Symmetric) Suppose $\varphi, \psi \in c(G; X)$ and $\varphi \overset{c}{\simeq} \psi$. Let $\Phi: G \to a^*(X \times I)$ be a c-homotopy between φ and ψ. Let $r: X \times I \to X \times I$ be defined by $r(x,t) = (x, 1-t)$ for each $(x,t) \in X \times I$. Define $\psi: G \to a^*(X \times I)$ by $\psi(v) = r(\Phi(v))$ for every $v \in G$. One can easily check that ψ is a c-homotopy between ψ and φ. Hence $\psi \overset{c}{\simeq} \varphi$.

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c) (Transitive) Suppose $\varphi,\psi,\chi \in c(G; X)$ and $\varphi \overset{c}{\simeq} \psi$ and $\psi \overset{c}{\simeq} \chi$. Let Φ and Ψ be c-homotopies between φ and ψ and between ψ and χ, respectively. Define maps $m:X \times I \to X \times [0,1/2]$ and $n:X \times I \to X \times [1/2,1]$ by $m(x,t) = (x,t/2)$ and $n(x,t)=(x,(1/2)+ + (t/2))$ for each $(x,t) \in X \times I$. Now, define $\Sigma:G \to a^*(X \times I)$ by $\Sigma(v) = m(\Phi(v)) \cup n(\Psi(v))$, for each $v \in G$. Clearly, Σ satisfies the condition (3.1)(i) and $\varphi(v) = p_X(\Sigma(v) \cap (X \times \{0\})$ and $\chi(v) = = p_X(\Sigma(v) \cap (X \times \{1\}))$ for every $v \in G$. Hence, Σ is a c-homotopy between φ and χ, and therefore $\varphi \overset{c}{\simeq} \chi$, provided we prove that Σ also satisfies the condition (3.1) (ii).

Let $(v,w) \in G \times G$ be a d-pair. Then $\overline{\Sigma(v)} \cap \Sigma(w) = = [\overline{m(\Phi(v)) \cup n(\Psi(v))}] \cap [m(\Phi(w)) \cup n(\Psi(w))] = A \cup B \cup C \cup D$, where $A = \overline{m(\Phi(v))} \cap m(\Phi(w))$, $B = \overline{m(\Phi(v))} \cap n(\Psi(w))$, $C = \overline{n(\Psi(v))} \cap m(\Phi(w))$, and $D = \overline{n(\Psi(v))} \cap n(\Psi(w))$. Since Φ and Ψ are c-functions and m and n are homeomorphisms, $A = D = \emptyset$. We claim that $B = \emptyset$ (and analogously, that $C = \emptyset$).

Indeed, suppose $B \neq \emptyset$. Then there is an $x \in X$ such that $(x,1/2) \in B$. It follows that $(x,1) \in \overline{\Phi(v)}$ and $(x,0) \in \Psi(w)$. Hence, $x \in p_X(\Psi(w) \cap (X \times \{0\})) = \Psi(w) = p_X(\Phi(w) \cap (X \times \{1\}))$ and $(x,1) \in \Phi(w)$. However, this is impossible because Φ is a c-function so that $\overline{\Phi(v)} \cap \Phi(w) = \emptyset$

For a c-function $\varphi:G \to a^*X$, let $[\varphi]$ denote the equivalence class of φ with respect to the relation $\overset{c}{\simeq}$ (called a c-homotopy class of φ) and let $c[G;X]$ denote the quotient set $c(G;X) / \overset{c}{\simeq}$.

Moreover, we can consider the association $(X,G) \mapsto \mapsto c[G; X]$ as a bifunctor $c:\tau^o \times G \to S$ provided we define c on morphisms as follows. Let $(f,m) \in Mor_{\tau^o G}((X,G),(Y,H))$. Then $c(f,m):c[G;X] \to c[H;Y]$ is given by $c(f,m)([\varphi]) = [\psi]$, where $\varphi \in c(G;X)$ and $\psi(w) = \cup_{v \in m^{-1}(w)} f^{-1}(\varphi(v))$ for $w \in H$.

The above notions are defined for pairs as follows.

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In an analogous fashion they can be also introduced for n-tuples.

Let X be a topological space and let Y be a subspace of X. Let G be a graph and let H be a subgraph of G. A c-function $\varphi: G \to a^*X$ is a c-function from (G,H) into (X,Y) (in notation, $\varphi: (G,H) \to (X,Y)$) provided $\varphi(v) \subset Y$ for every $v \in H$. Two c-functions $\varphi, \psi: (G,H) \to (X,Y)$ are c-homotopic provided there is a c-function $\Phi: (G,H) \to (X \times I, Y \times I)$ such that $\varphi(v) = p_X(\Phi(v) \cap (X \times \{0\}))$ and $\psi(v) = p_X(\Phi(v) \cap (X \times \{1\}))$ for every $v \in G$. One can prove that the relation of c-homotopy is an equivalence relation on the set $c(G,H; X,Y)$ of all c-functions from (G,H) into (X,Y). The equivalence class of a $\varphi \in c(G,H;X,Y)$ is denoted by $[\varphi]$ while $c[G,H;X,Y]$ denotes the set of all equivalence classes. Just like in the absolute case, we can consider the association $((X,Y),(G,H)) \mapsto c[G,H,X,Y]$ as a bifunctor $c_2: \tau_2^o \times G_2 \to S$.

When (X,Y) is the pair $(I^n, \dot{I}^n)$ and (G,H) is the pair (G,v), the set $c_n(G,v) = c[G,v; I^n, \dot{I}^n)$ becomes a group under the binary operation o defined as follows. Let $m,q: I^n \to I^n$ be defined by $m(t_1,\ldots,t_n) = (t_1/2, t_2, \ldots, t_n)$ and $q(t_1,\ldots,t_n) = ((1+t_1)/2, t_2, \ldots, t_n)$. Now, for $[\varphi], [\psi] \in c_n(G,v)$ put $[\varphi] o [\psi] = [X]$ where the c-function $\chi: (G,v) \to (I^n, \dot{I}^n)$ is given by the formula $\chi(w) = m(\varphi(w)) \cup q(\psi(w))$ for every $w \in G$.

4. R-HOMOTOPY VERSUS C-HOMOTOPY

The main results in this section (Theorem 4.1)) shows that the bifunctors c and r are naturally equivalent. It follows that r-homotopy and c-homotopy are two equivalent approaches in the investigation of properties of topological spaces using graphs and in the investigation of properties of graphs using topological spaces.

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THEOREM 4.1. *There is a natural equivalence* θ *between bifunctors* c *and* r.

P r o o f. In order to prove the theorem, for each topological space X and each graph G, we shall define a bijection $\theta = \theta(X,G) : c[G;X] \to r[X;G]$ such that for every morphism $(f,m) \in Mor_{\tau^o \times G}((X,G),(Y,H))$ the diagram

$$\begin{array}{ccc} c[G;X] & \xrightarrow{c(f,m)} & c[H;Y] \\ \downarrow{\scriptstyle \theta(X,G)} & & \downarrow{\scriptstyle \theta(Y,H)} \\ r[X;G] & \xrightarrow[r(f,m)]{} & r[Y;H] \end{array}$$

commutes. This will be accomplished in the following six lemmas.

DEFINITION OF $\theta(X,G)$: Let $[\varphi] \in c[G;X]$. For each $x \in X$ let $\varphi^{-1}(x) = \{v \in G \mid x \in \varphi(v)\}$. We can consider φ^{-1} as a function of X into the hyperspace aG of all non-empty subsets of G.

LEMMA 4.2. *Every selection* $f : X \to G$ *of the function* φ^{-1} *is an* r*-function.*

P r o o f. Recall that f is a selection for φ^{-1} provided $f(x) \in \varphi^{-1}(x)$ for every $x \in X$. Hence $x \in \varphi(f(x))$ for every $x \in X$.

Let $(v,w) \in G \times G$ be a d-pair. Since φ is a c-function, $\overline{\varphi(v)} \cap \varphi(w) = \emptyset$. But, $f^{-1}(v) \subset \varphi(v)$ and $f^{-1}(w) \subset \varphi(w)$ so that $\overline{f^{-1}(v)} \cap f^{-1}(w) = \emptyset$. Hence, f is an r-function.

LEMMA 4.3. *Every two selections* $f_1, f_2 : X \to G$ *of the function* φ^{-1} *are* r*-homotopic.*

P r o o f. Define a function $F : X \times I \to G$ as follows.

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$$F(x,t) = \begin{cases} f_1(x), & (x,t) \in X\times[0,1/2) \\ f_2(x), & (x,t) \in X\times[1/2,1] \end{cases}$$

We claim that F is an r-homotopy between f_1 and f_2.

In order to verify this claim it clearly suffices to show that F satisfies the condition (3.1)(ii).

Consider a d-pair $(v,w) \in G \times G$. Then $\overline{F^{-1}(v)} \cap F^{-1}(w) = A \cup B \cup C \cup D$, where $A = P \cap R$, $B = P \cap S$, $C = Q \cap R$, $D = Q \cap S$, $P = \overline{f_1^{-1}(v)} \times [0,1/2]$, $Q = \overline{f_2^{-1}(v)} \times [1/2,1]$, $R = f_1^{-1}(w) \times [0,1/2]$, and $S = f_2^{-1}(w) \times [1/2,1]$. Since f_1 and f_2 are r-functions, $A = D = \emptyset$. We claim that $B = \emptyset$ (and, analogously, that $C = \emptyset$).

Indeed, if $(x,1/2) \in B$, then $x \in \overline{f^{-1}(v)} \cap f^{-1}(w) \subset \overline{\varphi(v)} \cap \varphi(w)$. However, the intersection $\overline{\varphi(v)} \cap \varphi(w)$ is empty because φ is a c-function and (v,w) is a d-pair. Hence, $B = \emptyset$.

LEMMA 4.4. *Let $\varphi, \psi: G \to a^*X$ be c-functions and let $f: X \to G$ and $g: X \to G$ be selections of $\varphi^{-1}: X \to aG$ and $\psi^{-1}: X \to aG$, respectively. If $\varphi \overset{C}{\simeq} \psi$, then $f \overset{r}{\simeq} g$.*

P r o o f. Let $\Phi: G \to a^*(X \times I)$ be a c-homotopy between φ and ψ. Let $F: X \times I \to G$ be a selection for the function $\Phi^{-1}: X \times I \to aG$. We shall prove that $F_0: X \to G$ defined by $F_0(x) = F(x,0)$ (for $x \in X$) is a selection for φ^{-1}. Then it follows from the Lemma 4.3. that $f \overset{r}{\simeq} F_0$. Since one can similarly show that $g \overset{r}{\simeq} F_1$, it follows that $f \overset{r}{\simeq} g$.

Let $x \in X$. Since F is a selection for Φ^{-1}, $F(x,0) \in \Phi^{-1}(x,0)$. Hence, $(x,0) \in \Phi(F(x,0))$ and $x \in p_X(\Phi(F(x,0)) \cap (X \times \{0\})) = \varphi(F(x,0))$. Finally, we get $F_0(x) = F(x,0) \in \varphi^{-1}(x)$, i.e. that F_0 is a selection for φ^{-1}.

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It follows from the Lemmas 4.2.-4.4. that the function $\theta = \theta(X,G): c[G;X] \to r[X;G]$ which associates to a class $[\varphi] \in c[G;X]$ the r-homotopy class $[f]$ of any selection f of the function φ^{-1} is well-defined.

LEMMA 4.5. *The function* θ *is one-to-one.*

P r o o f. Consider $[\varphi], [\psi] \in c[G;X]$. Suppose $\theta([\varphi]) = \theta([\psi])$. We shall prove that $\varphi \overset{c}{\simeq} \psi$.

Let $f: X \to G$ be a selection of the function φ^{-1} and let $g: X \to G$ be a selection of the function ψ^{-1}. Let $H: X \times I \to G$ be an r-homotopy between f and g. Define functions $\chi: G \to a^*(X \times I)$, $\alpha: G \to a^*X$, and $\beta: G \to a^*X$ by $\chi(v) = H^{-1}(v)$, $\alpha(v) = p_X(\chi(v) \cap (X \times \{0\}))$, and $\beta(v) = p_X(\chi(v) \cap (X \times \{1\}))$ for every $v \in G$, respectively.

In order to show that φ and ψ are c-homotopic we shall prove that $\alpha \overset{c}{\simeq} \psi$ and $\beta \overset{c}{\simeq} \varphi$. Since χ is a c-homotopy between α and β, this would imply $\varphi \overset{c}{\simeq} \psi$.

Let a homeomorphism $m: X \times I \to X \times [0,1/3]$ be defined by $m(x,t) = (x,t/3)$ for each $(x,t) \in X \times I$. Next, we define a function $k : G \to a^*(X \times I)$ by $k(v) = m(\chi(v)) \cup (g^{-1}(v) \times [1/3,2/3] \cup (\psi(v) \times [2/3,1])$. We claim that k is a c-homotopy between α and ψ.

Indeed, let us check that the function k satisfies the condition (3.1)(ii). The other required properties of k are easy to verify.

Let $(v,w) \in G \times G$ be a d-pair. Then $\overline{k(v)} \cap k(w) = A \cup B \cup C \cup D \cup E \cup F \cup J$, where $A = P \cap P'$, $B = P \cap Q'$, $C = Q \cap P'$, $D = Q \cap Q'$, $E = Q \cap R'$, $F = R \cap Q'$, $J = R \cap R'$, $P = \overline{\psi(v)} \times [2/3,1]$, $Q = g^{-1}(v) \times [1/3,2/3]$, $R = \overline{m(\chi(v))}$, $P' = \psi(w) \times [2/3,1]$, $Q' = g^{-1}(w) \times [1/3,2/3]$ and $R' = m(\chi(w))$. Since ψ, g^{-1}, and χ are c-functions, $A = D = J = \emptyset$. We shall now prove that $B = \emptyset$ and $E = \emptyset$. One can analogously prove that $C = \emptyset$ and $F = \emptyset$. It follows that $\overline{k(v)} \cap k(w) = \emptyset$. Since a similar argument shows that $k(v) \cap \overline{k(w)} = \emptyset$, we get $\alpha \overset{c}{\simeq} \psi$. In an analogous fashion one can prove that $\beta \overset{c}{\simeq} \varphi$ and thus conclude the proof.

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Suppose that there is an $x \in X$ such that $(x,2/3) \in C$. Then $x \in g^{-1}(v)$ and $x \in \psi(w)$. But, $g^{-1}(v) \subset \psi(v)$ so that $x \in \overline{\psi(v)} \cap \psi(w)$. This is impossible because ψ is a c-function.

Finally, suppose that there is an $x \in X$ such that $(x,1/3) \in E$. Then $x \in \overline{g^{-1}(v)}$ and $(x,1) \in H^{-1}(w)$. Hence, $H(x,1) = g(x) = w$ so that $x \in \overline{g^{-1}(v)} \cap g^{-1}(w)$. However, this is impossible because g^{-1} is a c-function.

LEMMA 4.6. *The function* θ *is onto.*

P r o o f. Let $[h] \in r[X;G]$. Put $\varphi = h^{-1}: G \to a^*X$. Since $\varphi^{-1}(x) = \{h(x)\}$ for every $x \in X$, we see that h is a selection of φ^{-1}. This implies that $\theta([\varphi]) = [h]$.

LEMMA 4.7. *For every morphism* $(f,m) \in Mor_{\tau^o \times G}((X,G), (Y,H))$ *the equality* $\theta(Y,H) o c(f,m) = r(f,m) o \theta(X,G)$ *holds.*

P r o o f. Let $\psi \in c(G;X)$. Define $\chi \in c(H;Y)$ by $\chi(w) = U_{v \in m^{-1}(w)} f^{-1}(\psi(v))$ for $w \in H$. Let $g: Y \to G$ be a selection of $\psi^{-1}: Y \to aG$. It suffices to prove that $m o g o f$ is a selection of $\chi^{-1}: X \to aG$. Let $y \in Y$. Since g is a selection of ψ^{-1}, we have $f(y) \in \psi(g(f(y)))$ and $y \in f^{-1}[\psi(g(f(y)))]$. Hence, $y \in \chi[(m_o g_o f)]$. In other words, $(m_o g_o f)(y) \in \chi^{-1}(y)$ for every $y \in Y$.

REMARK 4.8. Only minor changes in the above proof are needed in order to prove the version of the Theorem 4.1 for pairs and n-tuples.

5. S-FUNCTIONS AND E-FUNCTIONS

Demaria [1] has also introduced the notion of a strongly regular function (or an s-function) as a function $f: X \to G$ of a topological space X into a graph G which satisfies the condition $\overline{f^{-1}(v)} \cap \overline{f^{-1}(w)} = \emptyset$ for every d-pair $(v,w) \in G \times G$.

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By replacing throughout the section 2 the letter "r" with the letter "s" we can define the notion of s-homotopy and the bifunctor $s: \tau^{o} \times G \to S$. This could also be done for pairs and more generally for n-tuples. Moreover, all statements about r-functions and r-homotopy hold also for s-functions and s-homotopy. In particular, as the following lemma shows, the relation of s-homotopy is an equivalence relation contrary to the claim in [2,p. 142]

LEMMA 5.1. *The relation* $\overset{s}{\simeq}$ *is an equivalence relation on the set* $s(X;G)$ *of all s-functions of* X *into* G.

P r o o f. We leave to the reader to check that $\overset{s}{\simeq}$ is reflexive and symmetric. In order to prove that it is transitive, suppose $f,g,h \in s(X;G)$, $f \overset{s}{\simeq} g$, and $g \overset{s}{\simeq} h$. Let F and G be s-homotopies between f and g and between g and h, respectively. Define a function $H: X \times I \to G$ by

$$H(x,t) = \begin{cases} F(x,3t), & (x,t) \in X \times [0,1/3] \\ g(x), & (x,t) \in X \times [1/3,2/3] \\ G(x,3t-2), & (x,t) \in X \times [2/3,1]. \end{cases}$$

Then H is an s-homotopy between f and h.

In analogy with the section 3, we shall say that a function $\varphi: G \to a^*X$ of a graph G into the hyperspace a^*X of all subsets of a topological space X is an e-function provided $X = \cup\{\varphi(v) \mid v \in G\}$ and $\overline{\varphi(v)} \cap \overline{\varphi(w)} = \emptyset$ for every d-pair $(v,w) \in G \times G$. The replacement of the letter "c" with the letter "e" in the section 3 gives us the notion of e-homotopy and the bifunctor $e: \tau^{o} \times G \to S$. Of course, this also applies to pairs and n-tuples. Moreover, only routine changes in the section 4 are needed to get the proof of the following result.

THEOREM 5.2. *The bifunctors* e *and* s *are naturally equivalent.*

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Related to the bifunctor e is the bifunctor b defined as follows. Let c*X denote the hyperspace of all closed subsets of a topological space X. A function $\varphi: G \to c^*X$ of a graph G into c*X is a b-function provided $X = \cup\{\varphi(v) \mid v \in G\}$ and $\varphi(v) \cap \varphi(w) = \emptyset$ for every d-pair $(v,w) \in G \times G$. Two b-functions $\varphi, \psi: G \to c^*X$ are b-homotopic (in notation, $\varphi \overset{b}{\simeq} \psi$) provided there is a b-function $\Phi: G \to c^*(X \times I)$ such that $\varphi(v) = p_X(\Phi(v) \cap (X \times \{0\}))$ and $\psi(v) = p_X(\Phi(v) \cap (X \times \{1\}))$ for every $v \in G$. It is clear now how we can define the bifunctor b and that there is a "b-version" of all of the section 3.

THEOREM 5.3. *The bifunctors* e *and* b *are naturally equivalent.*

P r o o f. For a $\varphi \in e(G;X)$, define $\varphi^* \in b(G;X)$ by $\varphi^*(v) = \overline{\varphi(v)}$. Then φ^* is a b-function and the formula $\Phi(v) = [\varphi(v) \times [0,1)] \cup [\varphi^*(v) \times \{1\}]$ for $v \in G$ defines an e-homotopy between φ and φ^*. It is now routine to show that $\tau = \tau(X;G): e[G;X] \to b[G;X]$ given by $\tau([\varphi]) = [\varphi^*]$ defines a natural equivalence between e and b.

The following corollary of the theorems 5.3 and 5.2 improves the main result in [3].

COROLLARY. *The bifunctors* b *and* e *are naturally equivalent.*

REMARK 5.5. Just like in the section 4, all of the results in this section hold also for pairs and n-tuples.

6. DIGRAPHS

In the case when G is a directed graph (or a digraph), Demaria considered o-regular and o*-regular functions of a space X into G. In order to define these classes of functions, we shall call a pair $(v,w) \in G \times G$ a j-pair provided either

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v = w or (v,w) is an edge of G. An ordered pair of elements of G which is not a j-pair is called a d-pair.

A function $f:X \to G$ is an or-function (o*r-function) provided $f^{-1}(v) \cap f^{-1}(w) = \emptyset$ ($f^{-1}(v) \cap f^{-1}(w) = \emptyset$) for every d-pair $(v,w) \in G \times G$. Of course, it is now possible to introduce or-homotopy and o*r-homotopy and bifunctors or and o*r. Also, we can define oc-functions and o*c-functions, corresponding homotopies, and bifunctors oc and o*c. The method of proof in the section 4 with minor alternations allows one to prove that oc is naturally equivalent to or and that o*c is naturally equivalent to o*r. A similar statement is also true for pairs and for n-tuples.

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REZIME

PRIRODNA EKVIVALENCIJA GRAFU ASOCIRANIH BIFUNKTORA

Uveden je metod za ispitivanje topoloških prostora i grafova pomoću podesnog pojma homotopije za neke klase presli-

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kavanja grafova u hiperprostore topoloških prostora. Pokazano je da je ovaj pristup prirodno ekvivalentan metodi Demaria koji je zasnovan na pojmu homotopije za neke klase preslikavanja topoloških prostora u grafove.

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ON RANDOM BEST APPROXIMATIONS

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ABSTRACT

In this paper we prove a random generalization of S. Reich's theorem on best approximations. As a corollary we obtain a result on random fixed points.

1. INTRODUCTION

In [8] S. Reich proved the following theorem on best approximations.

Theorem A. *Let S be a nonempty compact and convex subset of a Banach space* $(E,\|\|)$ *and* $F : S \to 2^E$ *(the family of all nonempty subsets of E) a continuous multifunction such that* $F(x)$ *is compact and convex for each* $x \in S$. *Then there exists an* $x \in S$ *such that:*

$$d(x,F(x)) = \inf_{(u,v) \in F(x)\times S} \|u - v\|.$$

V.M. Sehgal and S.P. Singh generalized Theorem A in the following way [10].

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Theorem B. *Let* $(E,\|\|)$ *be a Banach space,* (Ω,Σ) *a measurable space with* Σ *a* σ *algebra of subsets of* Ω *and* S *a nonempty compact and convex subset of* E. *If* $T : \Omega \times S \to 2^E$ *is a continuous random operator with compact and convex values, then there exists a measurable mapping* $g : \Omega \to S$ *satisfying:*

$$d(T(\omega,g(\omega)),g(\omega)) = \inf\{\|u - v\|, \quad (u,v) \in T(\omega,g(\omega)) \times S\}$$

for each $\omega \in \Omega$.

In [10], as a corollary of Theorem A, a result on random fixed points is obtained.

We shall prove in this paper a similar result as Theorem B, for mappings T with a stochastic domain C.

2. PRELIMINARIES

In the paper we shall use the following notations: by (Ω,Σ) we shall denote a σ finite complete measure space, $(E,\|\|)$ a separable Banach space, $2^E = \{X | X \subseteq E, X \neq \emptyset\}$, $2^E_k = \{X | X \in 2^E, X \text{ is compact}\}$ and $CB(E) = \{A | A \in 2^E, A \text{ is closed and bounded}\}$.

A mapping $C : \Omega \to 2^E$ is *weakly measurable* if and only if for all open $D \subseteq E$:

$$C^{-1}(D) = \{\omega | \omega \in \Omega, C(\omega) \cap D \neq \emptyset\} \in \Sigma.$$

If a mapping $C : \Omega \to 2^E$ is such that $C(\omega)$ is compact, for every $\omega \in \Omega$, then C is weakly measurable if and only if $C^{-1}(G) \in \Sigma$, for every closed subset G of E [6]. A mapping $C : \Omega \to 2^E$ is *separable* [4] if and only if C is weakly measurable and there exists a countable set $Z \subseteq E$ such that for all $\omega \in \Omega$:

$$\overline{Z \cap C(\omega)} = C(\omega).$$

If $C : \Omega \to 2^E$ then $GrC = \{(\omega,x) | (\omega,x) \in \Omega \times E, x \in C(\omega)\}$.

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Definition 1. [4] *Let* $C : \Omega \to 2^E$ *be a weakly measurable mapping. A mapping* $T : GrC \to 2^E$ *is called a random operator with a stochastic domain* C *if and only if for all* $x \in E$ *and open* $D \subseteq E$:

$$\{\omega | \omega \in \Omega,\ x \in C(\omega), T(\omega,x) \cap D) \neq \emptyset\} \in \Sigma$$

Definition 2. [4] *Let* $T : GrC \to 2^E$ *be a random operator with a stochastic domain* C. T *is called a continuous random operator if and only if for all* $\omega \in \Omega$, $T(\omega,\cdot)$ *is a continuous mapping from* $C(\omega)$ *into* $(CB(E),H)$, *where* H *is the Hausdorff metric on* $CB(E)$.

Theorem C. [3] *Let* $f : GrC \to R$ *be a continuous random function with a separable random domain* C *and* $a : \Omega \to R$ *a measurable function. Then the mapping* $F : \omega \mapsto F(\omega)$ *is measurable, where:*

$$F(\omega) = \{x | x \in C(\omega),\ f(\omega,x) \leqslant a(\omega)\}, \quad \omega \in \Omega.$$

Theorem D. [5] *Let* $C : \Omega \to 2^E$ *be a mapping with complete values. Then* C *is weakly measurable if and only if there exists a countable family* $U = \{u_1,u_2,\ldots\}$ *of measurable selectors for* C *such that:*

$$C(\omega) = \overline{U(\omega)} = \overline{\{u_n(\omega) | n \in N\}}.$$

Theorem E. [7] *If* $F : \Omega \to E$ *is weakly measurable with complete values, then* F *has a measurable selector.*

3. A THEOREM ON BEST APPROXIMATIONS

The proof of the next lemma is similar to the proof of Lemma 2 from [10].

Lemma. *Let* $C : \Omega \to 2^E_k$ *and* $F : GrC \to 2^E_k$ *be a conti-*

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nuous random operator. Then for every $\omega \in \Omega$ *the real functions:*

$$x \mapsto d(x,F(\omega,x)) \text{ and } x \mapsto d(C(\omega),T(\omega,x))$$

are continuous on $C(\omega)$.

Proof. From the inequality:

$$|d(x,F(\omega,x)) - d(y,F(\omega,y))| \leq D(F(\omega,x),F(\omega,y)) + \|x-y\|$$

for every $x,y \in C(\omega)$ and $\omega \in \Omega$ it follows that the mapping $x \mapsto d(x,F(\omega,x))$ is continuous on $C(\omega)$. Let for $\omega \in \Omega$:

$$g_\omega(x) = d(C(\omega),F(\omega,x)), \quad x \in C(\omega).$$

We shall prove the continuity of g_ω for every $\omega \in \Omega$.

Suppose that g_ω is not continuous at the point $x_0 \in C(\omega)$. Then there exists an $\varepsilon > 0$ and a sequence $\{x_n\}$ from $C(\omega)$ so that:

$$\lim_{n\to\infty} x_n = x_0 \quad \text{and} \quad |g(x_n) - g(x_0)| > \varepsilon, \ n \in \mathbb{N}.$$

We shall prove that this leads to a contradiction. Since $F(\omega,x_0)$ and $C(\omega)$ are compact, there exist $a \in F(\omega,x_0)$ and $b \in C(\omega)$ so that:

$$d(C(\omega),F(\omega,x_0)) = \|a - b\| = \inf_{(u,v) \in C(\omega) \times F(\omega,x_0)} \|u - v\|.$$

Since the mapping $x \mapsto F(\omega,x)$ is continuous on $C(\omega)$, it is a lower semicontinuous mapping which implies the existence of a sequence $\{y_{n_i}\}$, $y_{n_i} \in F(\omega,x_{n_i})$, such that $\lim_{i\to\infty} y_{n_i} = a$. Then we have that:

$$d(C(\omega),F(\omega,x_{n_i})) \leq \|b - y_{n_i}\| \leq \|b - a\| + \|a - y_{n_i}\| =$$

$$= d(C(\omega),F(\omega,x_0)) + \|a - y_{n_i}\|$$

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and so:

$$d(C(\omega),F(\omega,x_{n_i})) - d(C(\omega),F(\omega,x_0)) \leqslant \|a - y_{n_i}\|.$$

This implies that there exists $i_0(\varepsilon) \in \mathbb{N}$ so that $g(x_{n_i}) - g(x_0) < \varepsilon$, for every $i \geqslant i_0(\varepsilon)$. On the other hand let, for every $n \in \mathbb{N}$, $a_n \in C(\omega)$ and $y_n \in F(\omega,x_n)$ so that:

$$\|a_n - y_n\| = d(C(\omega),F(\omega,x_n)).$$

Since $C(\omega)$ is compact there exists a subsequence $\{a_{n_{i_r}}\}$ of $\{a_{n_i}\}$ such that $\lim_{r\to\infty} a_{n_{i_r}} = c$. Further, from the compactness of $F(\omega,x_0)$ and the uppersemicontinuity of $x \mapsto F(\omega,x)$ at $x = x_0$ there exists a subsequence $\{y_{n_{i_{r(k)}}}\}$ such that $\lim_{k\to\infty} y_{n_{i_{r(k)}}} = y \in F(\omega,x_0)$. Then we have that:

$$d(C(\omega),F(\omega,x_0)) \leqslant d(c,y) \leqslant d(c,a_{n_{i_{r(k)}}}) +$$

$$+ d(a_{n_{i_{r(k)}}},y_{n_{i_{r(k)}}}) + d(y_{n_{i_{r(k)}}},y) = d(c,a_{n_{i_{r(k)}}}) +$$

$$+ d(C(\omega),F(\omega,x_{n_{i_{r(k)}}})) + d(y_{n_{i_{r(k)}}},y).$$

This implies that

$$d(C(\omega),F(\omega,x_0)) - d(C(\omega),F(\omega,x_{n_{i_{r(k)}}})) < \varepsilon$$

for $k \geqslant k_0(\omega)$. Hence there exists $k(\varepsilon)$ so that

$$|g(x_{n_{i_{r(k)}}} - g(x_0)| < \varepsilon, \quad \text{for every } k \geqslant k(\varepsilon)$$

which is a contradiction.

Theorem. *Let $C : \Omega \to 2^E_k$ be a separable weakly measurable mapping such that $C(\omega)$ is convex, for every $\omega \in \Omega$, $T : GrC \to 2^E_k$ a continuous random operator such that $T(\omega,x)$ is convex for every $(\omega,x) \in GrC$. Then there exists a measurable*

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mapping $\varphi : \Omega \to E$ *such that for every* $\omega \in \Omega$, $\varphi(\omega) \in C(\omega)$ *and:*

$$d(\varphi(\omega),T(\omega,\varphi(\omega))) = d(C(\omega),T(\omega,\varphi(\omega))), \quad \omega \in \Omega.$$

Proof. Let $f : GrC \to \mathbb{R}$ be a real function defined by $f(\omega,x) = d(x,T(\omega,x)) - d(C(\omega),T(\omega,x))$, for every $\omega \in \Omega$ and $x \in C(\omega)$. Further, let for every $\omega \in \Omega$:

$$F(\omega) = \{x | x \in C(\omega), f(\omega,x) \leqslant 0\}.$$

Then $x \in F(\omega)$ implies that $f(\omega,x) = 0$, since for every $x \in C(\omega)$, $d(x,T(\omega,x)) \geqslant d(C(\omega),T(\omega,x))$. Hence, it remains to be proved that the mapping F has a measurable selector φ. In order to apply Theorem E we have to prove that the mapping F is measurable (it is obvious that $F(\omega)$ is closed and hence compact, for every $\omega \in \Omega$). We shall prove the measurability of F by using Theorem C, where $a(\omega) = 0$, for every $\omega \in \Omega$. Let us prove that f is a continuous random function. In [4] it is proved that the mapping $N_1 : GrC \to \mathbb{R}$, defined by

$$N_1(\omega,x) = d(x,T(\omega,x)), \quad (\omega,x) \in GrC$$

is a continuous random function. We shall prove that the mapping $N_2 : GrC \to \mathbb{R}$ defined by:

$$N_2(\omega,x) = d(C(\omega),T(\omega,x)), \quad (\omega,x) \in GrC$$

is a continuous random function as well. In fact, we shall prove that for every $r > 0$ and every $x \in E$:

$$\{\omega | x \in C(\omega), N_2(\omega,x) \leqslant r\} \in \Sigma.$$

For every $s > 0$ and every $x \in E$:

$$\{\omega | \omega \in \Omega, x \in C(\omega), N_2(\omega,x) < s\} =$$

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$$= \bigcup_{n\in\mathbb{N}} \{\omega|\omega \in \Omega, x \in C(\omega), d(u_n(\omega), T(\omega,x)) < s\}$$

where $C(\omega) = \overline{\bigcup_{n\in\mathbb{N}} u_n(\omega)}$ is the Castaing representation of C. Further [1], for every $n \in \mathbb{N}$:

$$\{\omega|\omega \in \Omega, x \in C(\omega), d(u_n(\omega), T(\omega,x)) < s\} \in \Sigma$$

and so

$$\{\omega|\omega \in \Omega, x \in C(\omega), N_2(\omega,x) < s\} \in \Sigma.$$

If Z is a countable subset of E such that $\overline{Z \cap C(\omega)} = C(\omega)$ it is easy to prove that:

$$\{\omega|\omega \in \Omega, x \in C(\omega), N_2(\omega,x) \leqslant r\} =$$

$$= \bigcap_{n\in\mathbb{N}} \bigcup_{z\in Z\cap L(x,1/n)} \{\omega|\omega \in \Omega, z \in C(\omega), N_2(\omega,z) < r+\tfrac{1}{n}\}$$

using the continuity of the mapping $d(C(\omega),T(\omega,\cdot))$ on $C(\omega)$, for every $\omega \in \Omega$. Hence, $\{\omega|\omega \in \Omega, x \in C(\omega), N_2(\omega,x) \leqslant r\} \in \Sigma$.

If φ is a measurable selector of F we have that $d(\varphi(\omega),T(\omega,\varphi(\omega))) = d(C(\omega),T(\omega,\varphi(\omega)))$, for every $\omega \in \Omega$.

Corollary. Let C and T be as in Theorem and $\overline{T(\omega,C(\omega))} \subseteq C(\omega)$, for every $\omega \in \Omega$. Then there exists a measurable mapping $\varphi : \Omega \to E$ so that $\varphi(\omega) \in C(\omega)$, for every $\omega \in \Omega$ and $\varphi(\omega) \in T(\omega,\varphi(\omega))$ (i.e. φ is a random fixed for the mapping T).

Proof. Since in this case we have that

$$d(C(\omega),T(\omega,\varphi(\omega))) = 0, \quad \text{for every } \omega \in \Omega$$

we have that $d(\varphi(\omega),T(\omega,\varphi(\omega))) = 0$ for every $\omega \in \Omega$.

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REZIME

O STOHASTIČKOJ NAJBOLJOJ APROKSIMACIJI

U ovom radu dokazana je stohastička generalizacija teoreme S. Reicha o najboljoj aproksimaciji. Kao posledica dobijen je rezultat o stohastičkim nepokretnim tačkama.

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ZERO-CONTINUOUS FUNCTIONS ON K-, N-, AND COMPLETE SPACES

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ABSTRACT

Zero-continuous functions with the help of the family of all the summable sequences - Theorem 3.1. and with its subfamilies - Theorem 3.4. and Theorem 3.6. are characterized. These results made it possible to obtain the continuity of summable and supersummable continuous functions.

1. INTRODUCTION

In this paper we shall investigate zero-continuous functions (continuous at x = 0) defined on the convergence group and with the values in a convergence group, using the family of all the summable sequences and some of its subfamilies. We shall obtain as consequences some results from [4] and [8] on the comparison of K-, N- and complete convergences.

We shall introduce the notions of supersummable and summable continuous functions as a generalization of the corresponding notions from [2], [4] and [5]. We shall obtain some

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characterizations of such functions.

2. PRELIMINARY NOTIONS AND NOTATIONS

Let X be a non empty set. The convergence on X is a function $G: X^N \to 2^X$ or equivalently G is a subset of $X^N \times X$. If $x \in G((x_n))$, we shall write $x_n \to x(G)$.

We shall use in the paper the following conditions for a convergence ([1],[7]):

(S) For each $x \in X$, and $x_n = x (n \in N)$, $x_n \to x(G)$.

(F) If $x_n \to x(G)$, then $y_n \to x(G)$ for any subsequence (y_n) of (x_n).

(H) If $x_n \to x(G)$ and $x_n \to y(G)$, then $x = y$.

(U) If $x_n \not\to x(G)$, then there exists a subsequence of (x_n) such that $z_n \not\to x(G)$ for every subsequence (z_n) of (y_n).

Let X be an additive group.

(L) If $x_n \to x(G)$ and $y_n \to y(G)$, then $x_n - y_n \to x - y(G)$.

(K) If $x_n \to 0(G)$, then each subsequence (y_n) of (x_n) has a subsequence (z_n) such that $\sum_{k=1}^{n} z_k \to z(G)$ for some $z \in X$.

(N) If $x_n \to 0(G)$, then each subsequence of (x_n) has a subsequence (y_n) such that $\sum_{k=1}^{n} z_k \to z(G)$ for every subsequence (z_n) of (y_n) and some $z \in X$.

In papers [3] and [6] normed K- and N-spaces were constructed, respectively, which were not complete spaces.

Let $S(G)$ be the family of all summable sequences with respect to the convergence G on a group X, i.e. $(x_n) \in S(G)$ iff $\sum_{k=1}^{n} x_k \to x(G)$ for some $x \in X$.

Let $S_s(G)$ be the family of all subseries summable sequences with respect to the convergence G, i.e. $(x_n) \in S_s(G)$ means that each subsequence (x_{n_k}) of (x_n) is summable with respect to the convergence G.

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3. ZERO-CONTINUOUS FUNCTIONS

Let X_1 and X_2 be two groups endowed with the convergences G_1 and G_2, respectively, such that they satisfy conditions S and L. A map f of X_1 into X_2 is called (sequentially) continuous at a point $x \in X_1$ if for any sequence (x_n) in X_1 such that $x_n \to x(G_1)$ holds $f(x_n) \to f(x)(G_2)$. A function $f: X_1 \to X_2$ is continuous at X_1 (or simply continuous), if for any $x \in X_1$ and for any sequence (x_n) in X_1 such that $x_n \to x(G_1)$ holds $f(x_n) \to f(x)(G_2)$.

We shall consider in this paper only those functions f which satisfy condition $f(0) = 0$.

Since the convergence G_1 satisfies conditions L and S, the preceding definition of a continuous function at a point $x \in X_1$ implies for the functions with the property $f(0) = 0$ that for any sequence (y_n) in X_1 such that $y_n \to 0(G_1)$ holds $f(y_n) \to 0(G_2)$. The last condition is equivalent to the continuity of f at $x = 0$ (zero-continuous function).

If a function $f: X_1 \to X_2$, $f(0) = 0$, is additive, then f is continuous, iff it is zero-continuous. But there exist also non-additive functions f, $f(0) = 0$, which are continuous iff they are zero-continuous. For example, a quasi-norm q on a group, i.e. functional $q: X \to R$ such that $q(0) = 0$, $q(-x) = q(x)$ and $q(x+y) \leq q(x) + q(y)$. This follows from the inequality

$$|q(x+y) - q(x)| \leq q(y).$$

THEOREM 3.1. *Let* f *be a map from an FLSK convergence group* (X_1, G_1) *into a LSU convergence group* (X_2, G_2) *such that* $f(0) = 0$. *The function* f *is zero-continuous iff* $f(x_n) \to 0(G_2)$ *for any* $(x_n) \in S(G_1)$.

PROOF. Suppose that the theorem is not true. Then there exists a sequence (x_n) from X_1 such that $x_n \to 0(G_1)$ and $f(x_n) \not\to 0(G_2)$. Since G_2 satisfies condition U, there exists a subsequence (y_n) of (x_n) such that for each subsequence (z_n) of (y_n) it holds that $f(z_n) \not\to 0(G_2)$. We have by the F property

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of convergence G_1 $y_n \to 0(G_1)$. Since G_1 is a K-convergence, there exists a subsequence (v_n) of (y_n) such that $(v_n) \in S(G_1)$. By the assumption of theorem $f(v_n) \to 0(G_2)$. A contradiction.

Taking specially the identity map on a group, we obtain as a consequence the following

COROLLARY 3.2. ([Theorem 1. in 8]). *Let* G_1 *and* G_2 *be two SL convergences on an additive group* X *such that* G_1 *satisfies the Urysohn property*-U *and* G_2 *is a* K-*convergence with* F *property. If* $S(G_1) \supset S(G_2)$, *then* $G_1 \supset G_2$.

COROLLARY 3.3. ([4, Proposition 3.3.28.a.d]). *Let* u *be a homomorphism of Hausdorff topological* R-*modules of* G *into* H. *If* G *is metrizable and complete, then the following assertions are equivalent:*

a) u *is continuous;*

d) $\lim_{n\to\infty} u(x_n) = 0$ *for any summable sequence* (x_n) *in* G.

If we make more restrictive assumptions on convergences, then we can characterize the zero-continuity with a subfamily of the family $S(G_1)$.

THEOREM 3.4. *Let* f *be a map from an FLSN convergence group* (X_1, G_1) *into an LSU convergence group* (X_2, G_2), *such that* $f(0) = 0$. *The function* f *is zero-continuous iff* $f(x_n) \to 0(G_2)$ *for any* $(x_n) \in S_s(G_1)$.

The proof is analogous to the proof of Theorem 1.

COROLLARY 3.5. ([8, Theorem 2]). *Let* G_1 *and* G_2 *be two* N-*convergences on a group* X *which are topological. If* $S_s(G_1) = S_s(G_2)$ *then* $G_1 = G_2$.

Let X be a Hausdorff topological group which is metrizable. Let d be an invariant metric on X generating its topology. We denote by $S_a(G)$ the family of all absolutely summable sequences (x_n), i.e. such that

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$$\sum_{n=1}^{\infty} d(x_n,0) < \infty$$

and (x_n) is summable.

THEOREM 3.6. *Let* f *be a map from a Hausdorff topological group* X_1 *into a Hausdorff topological group* X_2 *such that* $f(0) = 0$. *If* X_1 *is metrizable and complete, then function* f *is zero-continuous iff* $f(x_n) \to 0$ *as* $n \to \infty$ *for any* $(x_n) \in S_a(G_1)$.

PROOF. Suppose that the theorem is not true. Then there exists a sequence (x_n) from X_1 such that $d(x_n,0) \to 0$ as $n \to \infty$ and $q(f(x_n)) \not\to 0$ as $n \to \infty$ for some quasi-norm q which belongs to the family of quasi-norms which generates the topology of X_1. Then there exists a subsequence (y_n) of (x_n), such that for each subsequence (z_n) of (y_n) holds $q(f(z_n)) \not\to 0$ as $n \to \infty$. There exists a subsequence (u_n) of (y_n) such that $d(u_n,0) < 2^{-n}$ $(n \in N)$. Since X_1 is complete, the sequence (u_n) is summable and so $(u_n) \in S_a(G_1)$. Then by the assumption of the Theorem $q(f(u_n)) \to 0$ as $n \to \infty$, which is the expected contradiction.

COROLLARY 3.7. ([8, Theorem 3]). *Let* G_1 *and* G_2 *be two convergences generated by two Banach space norms, respectively. If* $S_a(G_1) \supset S_a(G_2)$, *then* $G_1 = G_2$.

4. SUMMABLE CONTINUOUS FUNCTIONS

DEFINITION 4.1. *A function* $f: X_1 \to X_2$ *is (super-) summable continuous if for each sequence* $(x_n) \in S(G_1)$ $((x_n) \in S_s(G_1))$, i.e. $\sum_{k=1}^{n} x_k \to x(G_1)$ *for some* $x \in X_1$, *hold*

$$(f(x_n)) \in S(G_2) \quad \text{and} \quad \sum_{k=1}^{n} f(x_k) \to f(x)(G_2).$$

REMARK. Super-summable continuous maps were considered in [?, p. 819] for Hausdorff locally convex spaces under the name subcontinuous functions, in [5, Definition 8.6] on

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Hausdorff commutative groups under the name Σ-continuous functions and in [4, Definition 3.3.18] as a member of the family $P^c(X_1,X_2)$ where X_1 and X_2 are Hausdorff topological R-modules.

PROPOSITION 4.2. *Any super-summable continuous function is additive. If X_1 and X_2 are two convergence real vector spaces, then any super-summable continuous function is linear.*

PROOF. The first part of the Proposition is obvious. The proof of the second part of the Proposition is analogous to the proof of Proposition 3.3.19. from [4].

It is obvious that any continuous additive function is summable continuous and that any summable continuous function is super-summable continuous.

THEOREM 4.3. *Let X_1 be a group endowed with an FLSK convergence G_1 and X_2 be a group endowed with an FLUS convergence G_2. Then a summable continuous function f from X_1 into X_2 is a continuous additive function.*

PROOF. We have by Definition 4.1. for any $(x_n)\in S(G_1)$

$$f(x_n) = \sum_{k=1}^{n} f(x_k) - \sum_{k=1}^{n-1} f(x_k) \to 0 \text{ as } n \to \infty.$$

Then the assertion of Theorem 4.3. follows by Theorem 1.

We obtain in the same way, now with Theorem 3.4.

THEOREM 4.4. *Let X_1 be a group endowed with an FLSN convergence and X_2 be a group endowed with an FLUS convergence G_2. Then a super-summable continuous function f from X_1 into X_2 is a continuous additive function.*

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REZIME

NULA NEPREKIDNE FUNKCIJE NAD K-, N-, I KOMPLETNIM PROSTORIMA

U radu se ispituju funkcije koje su neprekidne (sekvencijalno) u tački x = 0 a koje su definisane nad konvergentnim grupama i sa vrednostima u konvergentnoj grupi. Ove tzv. nula-neprekidne funkcije ispituju se pomoću familije svih sumabilnih nizova - Teorema 3.1. i sa njenim podfamilijama - Teoreme 3.4. i 3.6. Uvode se i ispituju i sumabilno i super-sumabilno neprekidne funkcije.

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A NOTE ON SOME APPLICATIONS OF BELLMAN'S RESULTS ON ALMOST ORTHOGONAL SEQUENCES

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ABSTRACT

In this paper an asymptotic version of the Riesz-Fisher theorem is proved. Convergence in the square mean is connected with the almost orthogonal sequences.

The following Riesz-Fisher theorem is well-known: Let $\{g_n\}$ be a sequence of orthogonal functions in $L_2(a,b)$, and $\{b_n\}$ be a sequence of numbers from ℓ_2, i.e.

$$\sum_{n=1}^{\infty} |b_n|^2 < \infty ,$$

then there exists a function $f \in L_2(a,b)$ such that

$$f = \sum_{n=1}^{\infty} b_n g_n \quad \text{in} \quad L_2(a,b)$$

and

$$b_n = \int_a^b f \bar{g}_n \, dx.$$

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We shall prove in this note the following asymptotic version of the Riesz-Fisher theorem.

Theorem 1. *Let* $\{F_n\}$ *be a sequence of functions from the space* $L_2(a,b)$ *(where* $[a,b]$ *is a finite real interval) which does not converge in the square mean to zero. If*

$$\lim_{j\to\infty} \int_a^b F_i(x)\,\overline{F_j(x)}\,dx = 0 \quad (i \in \mathbb{N}),$$

then there exists a subsequence $\{g_k\}$ *of* $\{F_n\}$ *such that for any sequence* $\{b_k\}$ *from* ℓ_2, *there exists a function* $f \in L_2(a,b)$, *such that*

$$\sum_{k=1}^{\infty} \left| b_k - \frac{1}{\|g_k\|} \int_a^b f(x)\,\overline{g_k(x)}\,dx \right|^2 < \infty .$$

The main tool in the proof of the preceding Theorem 1 will be the following R. Bellman's theorem from [4].

Bellman theorem. *Let* $\{g_n\}$ *be on almost orthogonal sequence of functions on a finite interval* (a,b), *i.e.*

$$\int_a^b |g_n|^2\,dx = 1$$

and

$$\sum_{m,n} |a_{mn}|^2 < \infty$$

where

$$a_{mn} = \begin{cases} \int_a^b g_n(x)\overline{g_m(x)}dx & n \neq m \\ 0 & n = m. \end{cases}$$

If a sequence $\{b_n\}$ *is from* ℓ_2, *then there exists a function* $f \in L_2(a,b)$ *such that*

$$\sum_{k=1}^{\infty} \left| b_k - \int_a^b f(x)\overline{g_k(x)}dx \right|^2 \le \Big(\sum_{n,m} |a_{nm}|^2 \Big)\Big(\sum_{k=1}^{\infty} |b_k|^2 \Big).$$

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We shall use in the proof of Theorem 1, also an elementary theorem from [1], which is of a similar nature as Theorem 1 from [2], see also the Basic Matrix Theorem 2 ([3], p. 7-10).

Diagonal theorem. *Let* $[x_{ij}]$ $(i,j \in \mathbb{N})$ *be a matrix of nonnegative real numbers such that*

$$\lim_{j\to\infty} x_{ij} = 0 \quad (i \in \mathbb{N}),$$

$$\lim_{i\to\infty} x_{ij} = 0 \quad (j \in \mathbb{N})$$

and

$$\lim_{i\to\infty} x_{ii} = 0.$$

Then there exists an infinite set I, $I \subset \mathbb{N}$, *such that*

$$\sum_{i,j\in I} x_{ij} < \infty.$$

Proof of Theorem 1. Let

$$h_n = \frac{F_n}{\|F_n\|},$$

where

$$\|F_n\| = \left(\int_a^b |F_n|^2\, dx\right)^{\frac{1}{2}} \quad (n \in \mathbb{N}).$$

Since the sequence $\{F_n\}$ does not converge in the square mean to zero, there exist a number $m > 0$ and a subsequence $\{F_{n_k}\}$ such that $\|F_{n_k}\| \geq m > 0$ $(k \in \mathbb{N})$.

Hence

$$\left|\int_a^b h_{n_i}\, \bar{h}_{n_j}\, dx\right| \leq \frac{1}{m^2}\left|\int_a^b F_{n_i}\, \bar{F}_{n_j}\, dx\right|.$$

Let

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$$x_{ij} = \int_a^b h_{n_i} \bar{h}_{n_j} \, dx \quad \text{for} \quad i \neq j \quad \text{and} \quad x_{ii} = 0.$$

Then, we obtain

$$|x_{ij}|^2 \to 0 \quad \text{for} \quad j \to \infty, \ (i \in N),$$

and

$$|x_{ij}|^2 \to 0 \quad \text{for} \quad i \to \infty \quad (j \in N).$$

So we can apply the diagonal theorem on $|x_{ij}|^2$ $(i,j \in N)$. Then there exists an increasing sequence $\{p_i\}$ of natural numbers such

$$\sum_{i,j} |x_{p_i p_j}|^2 < \infty .$$

Let $\{g_k\}$ be the subsequence of the sequence $\{F_n\}$ with the sequence of indexes $\{n_{p_k}\}$. Then we have

$$\sum_{\substack{i,j \\ i \neq j}} \frac{1}{\|g_i\|^2 \|g_j\|^2} \left| \int_a^b g_i \bar{g}_j \, dx \right|^2 < \infty .$$

This means, that $\{g_k / \|g_k\|\}$ is an almost orthogonal sequence. Then by Bellman's theorem we obtain the conclusion of the theorem.

For example, the sequence of functions $\{F_n\}$, where $F_n(x) = \exp(i\lambda_n x)$ and $\{\lambda_n\}$ is sequence of real numbers such that $|\lambda_n| \to \infty$, satisfies the preceding theorem.

Remark. Combining the Riesz-Fisher theorem, Bellman's theorem and Theorem 1., we have the following: Let $\{F_n\}$ be a sequence of functions from the space $L_2(a,b)$ such that

$$\int_a^b |F_n|^2 dx = 1 \quad \text{and} \quad \lim_{j \to \infty} \int_a^b F_i(x) \overline{F_j(x)} \, dx = 0 \ (i \in N).$$

Let $\{b_n\}$ be an arbitrary element from ℓ_2. Then there exists a

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function f from $L_2(a,b)$ such that either $F = \sum b_n F_n$ and

$$b_n = \int_a^b f\,\bar{F}_n\,dx \quad \text{or} \quad b_n \neq \int_a^b f\,\bar{F}_n\,dx$$

and either the series

$$\sum_{k=1}^{\infty} \left|b_k - \int_a^b f(x)\,\overline{F_k(x)}\,dx\right|^2$$

is convergent or divergent and in this second case, there exists a subsequence $\{g_k\}$ of $\{F_k\}$ for which the preceding series is finite.

Almost orthogonal sequences have another interesting property which is included in the next theorem.

Theorem 2. *Let* $\{g_n\}$ *be an almost orthogonal sequence on interval* $[a,b]$ *and* $f \in L_2(a,b)$. *If* $\{f_n\}$ *is a sequence from* $L_2(a,b)$ *which converges in the square mean to* f, *then*

$$\lim_{n\to\infty} \sum_{k=1}^{\infty} \left(\int_a^b f_n(x)\overline{g_k(x)}dx\right)^2 = \sum_{k=1}^{\infty} \left(\int_a^b f(x)\overline{g_k(x)}dx\right)^2.$$

The proof is based on the following Bellman's result from [4].

Generalized Bessel inequality

If $\{g_n\}$ is an almost orthogonal sequence on the interval $[a,b]$ and $f \in L_2(a,b)$, then we have

$$\sum_{k=1}^{\infty} \left|\int_a^b f(x)\overline{g_k(x)}dx\right|^2 \leq \left[1 + \left(\sum_{m,n} |a_{mn}|^2\right)^{\frac{1}{2}}\right] \int_a^b |f(x)|^2 dx.$$

Proof of Theorem 2. We obtain by the generalized Bessel inequality

$$\sum_{k=1}^{\infty} \left|\int_a^b f_n(x)\overline{g_k(x)}dx \pm \int_a^b f(x)\overline{g_k(x)}dx\right|^2 \leq C \int_a^b |f_n \pm f|^2 dx,$$

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where,

$$C = 1 + \Big(\sum_{n,m} |a_{nm}|^2\Big)^{\frac{1}{2}}.$$

So we have

$$\Big|\sum_{k=1}^{\infty}\Big(\int_a^b f_n(x)\overline{g_k(x)}dx\Big)^2 - \sum_{k=1}^{\infty}\Big(\int_a^b f(x)\overline{g_k(x)}dx\Big)^2\Big|^2 \le$$

$$\le \sum_{k=1}^{\infty}\Big|\int_a^b f_n(x)\overline{g_k(x)}\,dx -$$

$$-\int_a^b f(x)\overline{g_k(x)}dx\Big|^2 \cdot \sum_{k=1}^{\infty}\Big|\int_a^b f_n(x)\overline{g_k(x)}dx +$$

$$+\int_a^b f(x)\overline{g_k(x)}dx\Big|^2 \le C^2\int_a^b |f_n-f|^2\,dx\int_a^b |f_n+f|^2\,dx \le$$

$$\le C^2\int_a^b |f_n-f|^2dx\,\Big\{\Big[\int_a^b |2f|^2dx\Big]^{\frac{1}{2}} + \Big[\int_a^b |f_n-f|^2dx\Big]^{\frac{1}{2}}\Big\}.$$

Letting $n \to \infty$, we obtain the assertion.

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REZIME

BELEŠKA O NEKIM PRIMENAMA BELLMANOVIH
REZULTATA O SKORO ORTOGONALNIM NIZOVIMA

Na osnovu rezultata R. Bellmana [4] i Dijagonalne teoreme iz [1] dokazana je asimptotska verzija Riesz-Fisherove teoreme - Teorema 1.

U teoremi 2. je povezana konvergencija u srednjem kvadratnom sa skoro ortogonalnim nizovima.

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ÜBER ZUFÄLLIGE FIXPUNKTE BEI STETIGEN ZUFÄLLIGEN OPERATOREN IN METRISCHEN VEKTORRÄUMEN

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ZUSAMMENFASSUNG

In der vorliegenden Arbeit wird ein Homotopieerweiterungsatz für zufällige Operatoren bewiesen. Er verallgemeinert ein entsprechendes deterministisches Resultat aus [7] und erlaubt die Herleitung von verschiedenen Fixpunktaussagen für zufällige Operatoren, ohne ständig die dazugehörigen deterministischen Aussagen zu verwenden.

1. EINLEITUNG

Fixpunktaussagen für zufällige Operatoren gehen auf Arbeiten der Prager Schule der Wahrscheinlichkeitstheorie um Spacek und Hanš zurück (s.z. B. [8]). Derartige Aussagen sind nützliche Hilfsmittel bei der Untersuchung von zufälligen Differential - bzw. Integralgleichungen (s.z. B. [1]). Seit dem Artikel von Bharucha-Reid 1976 ([2]) sind sehr viele Arbeiten über zufällige Fixpunkte erschienen. Zunächst übertrug man eine Reihe bekannter Fixpunktsätze auf die stochastische Situation, wobei die deterministi-

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Kennwörter: Zufällige Operatoren, zufällige Fixpunkte, meßbare Räume, meßbare (mengenwertige) Abbildungen, Homotopie von kompakten Abbildungen.

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schen Ergebnisse verwendet wurden. Neuere Fixpunktaussagen sichern die Lösbarkeit des stochastischen Fixpunktproblems unter der Voraussetzung der Lösbarkeit der zugehörigen deterministischen Probleme ganz allgemein (vgl. Satz 1 aus Abschnitt 2).

Wir wollen in unserer Note eine Möglichkeit andeuten, wie man auf der Grundlage eines zufälligen Fixpunktsatzes vom Schauder-Typ verschiedenartige allgemeine zufällige Fixpunktaussagen herleiten kann, ohne ständig das zugehörige deterministische Problem verwenden (und kennen) zu müssen. Dadurch ist eine Behandlung von Fixpunktproblemen innerhalb der stochastischen Theorie möglich. Hauptinstrument dafür ist ein Homotopieerweiterungssatz für zufällige kompakte Operatoren (Satz 4 aus Abschnitt 3). Er verallgemeinert einen Satz des Verfassers aus [7] durch die Erweiterung des Gültigkeitsbereiches auf zufällige Operatoren. Anwendungen des Homotopieerweiterungssatzes werden in Abschnitt 4 vorgestellt.

Die Ausführungen sollen nur andeuten, daß der von Granas [5] begründete Weg zur Leray-Schauder-Theorie auch für den Fall von zufälligen Operatorengleichungen gangbar ist. Dabei verzichten wir auf eine größte Allgemeinheit.

So betrachten wir hier nicht Operatoren mit zufälligem Definitionsbereich. Ferner beschränken wir uns auf kompakte zufällige Operatoren, obwohl die Sätze auch für kondensierende zufällige Operatoren bewiesen werden könnten.

Im folgenden stellen wir einige Begriffe und Bezeichnungen zusammen.

In der gesamten Arbeit bezeichne (Ω,γ,μ) einen vollständigen σ-finiten Maßraum.

Sei X ein metrischer Raum, so sei Cl(X) das System aller nichtleerer, abgeschlossener Teilmengen von X. Eine (mengenwertige) Abbildung $A : \Omega \to Cl(X)$ heißt meßbar (schwach meßbar), wenn für jede abgeschlossene (offene) Teilmenge M von X das vollständige Urbild $A^{-1}(M) = \{w \in \Omega : A(w) \cap M \neq \emptyset\}$ meßbar ist. Wir bemerken, daß im Falle kompakter Werte Meßbarkeit und schwache Meßbarkeit äquivalent sind. Speziell gilt das also für (punktwertige) Abbildungen $F : \Omega \to X$. Ist X ein vol-

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lständiger, separabler metrischer Raum, so ist ebenfalls eine Abbildung $A : \Omega \to Cl(X)$ meßbar genau dann, wenn sie schwach meßbar ist. In diesem Fall sind beide Meßbarkeitsbegriffe auch gleichwertig mit der Meßbarkeit des Graphen $Gr(A) := \{(w,x) \in \Omega \times X : x \in A(w)\}$, d. h., mit der Beziehung $Gr(A) \in \mathcal{A}(\gamma \times \mathcal{L}(X))$.
Dabei ist $\gamma \times \mathcal{L}(X)$ die von allen Mengen $S \times B$ ($S \in \gamma$, $B \in \mathcal{L}(X)$, $\mathcal{L}(X)$ System aller Borelmengen von X) erzeugte σ-Algebra. (vgl. [9], Theorem 3.5). Sei Y ein metrischer Raum, Ω und X wie oben erklärt.
Eine Abbildung $F : \Omega \times X \to Y$ heißt ein zufälliger Operator, wenn $F(\cdot,x)$ meßbar für jedes $x \in X$ ist. Der zufällige Operator $F : \Omega \times X \to Y$ heißt stetig (kompakt), wenn die Abbildung $F(w,\cdot)$ für jedes feste w stetig (kompakt) ist. Sei nun $X \subseteq Y$. Unter einem zufälligen Fixpunkt des zufälligen Operators $F : \Omega \times X \to Y$ verstehen wir eine meßbare Abbildung $x : \Omega \to X$ mit $F(w,x(w)) = x(w)$ für alle $w \in \Omega$.

2. EIN ALLGEMEINER ZUFÄLLIGER FIXPUNKTSATZ

Wir geben zunächst einen allgemeinen zufälligen Fixpunktsatz an, der das stochastische Problem vollständig auf ein deterministisches Problem zurückführt. Der Satz ist prinzipiell bekannt, er wurde für separable Banachräume und zufällige Definitionsbereiche von Engl [4] erstmals bewiesen (siehe auch [3], [13], [14]).
Wir geben eine Formulierung für metrische Räume an.

Satz 1. *Sei Y ein vollständiger, separabler metrischer Raum, X eine abgeschlossene Teilmenge von Y sowie $F : \Omega \times X \to Y$ ein stetiger zufälliger Operator. Für jedes $w \in \Omega$ habe die Abbildung $F(w,\cdot)$ einen Fixpunkt. Dann hat F einen zufälligen Fixpunkt.*

Beweis. Sei $A(w) := \{x \in X : x = F(w,x)\}$ $(w \in \Omega)$. Nach Voraussetzung gilt $A(w) \neq \emptyset$ $(w \in \Omega)$ und $A(w)$ ist für jedes $w \in \Omega$ abgeschlossene Teilmenge von X, da F stetig bezü-

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glich x und X abgeschlossen ist.
Wir zeigen nun die schwache Meßbarkeit der Abbildung $A : \Omega \to Cl(X)$. Sei dazu d die Metrik in Y und $f(w,x) := d(x,F(w,x))$ $(w \in \Omega, x \in X)$. Da $F(\cdot,x)$ für jedes feste $x \in X$ meßbar ist, folgt aus Theorem 3.5 in [9] die Meßbarkeit von $f(\cdot,x)$ für jedes $x \in X$. Da ferner $f(w,\cdot)$ stetig ist für jedes feste $w \in \Omega$, können wir Theorem 6.1 aus [9] auf f anwerden und $f : \Omega \times X \to \mathbb{R}$ ist damit $(\gamma \times \mathcal{L}(X))$-meßbar. Speziell gilt daher $f^{-1}(\{0\}) \in \gamma \times \mathcal{L}(X)$. Nun ist

$$Gr(A) = \{(w,x) : x \in A(w)\} = \{(w,x) : x = F(w,x)\} =$$

$$= \{(w,x) : f(w,x) = 0\} = f^{-1}(\{0\}) \in \gamma \times \mathcal{L}(X).$$

Daher hat A einen meßbaren Graphen und A ist somit schwach meßbar. Nach dem Selections-Theorem von Kuratowski und Ryll-Nardzewski [12] hat A einen meßbaren Selektor, das heißt, es existiert eine meßbare Abbildung $x : \Omega \to X$ mit $x(w) \in A(w)$ für jedes $w \in \Omega$. Damit gilt $x(w) = F(w,x(w))$ $(w \in \Omega)$ und $x : \Omega \to X$ ist zufälliger Fixpunkt für F.

Bemerkung. Wenden wir im Beweis von Satz 1 auf nicht das zitierte Selektionstheorem aus [12], sondern Theorem 5.6 aus [9] an, so erhalten wir wie in [4] sogar die Existenz einer höchstens abzählbaren Menge $\{x_1, x_2, \ldots\}$ von zufälligen Fixpunkten von F derart, daß für jedes $w \in \Omega$ die Menge $\{x_n(w)\}$, $n \in N$ dicht in $\{x \in X : x = F(w,x)\}$, also in der Menge aller Fixpunkte von $F(w,\cdot)$ liegt.
Realisierung von Satz 1 an (Satz 3). Eine naheliegende Verallgemeinerung des bekannten stochastischen Analogons zum Fixpunktsatz von Schauder (s.z. B.[2]) formulieren wir zur Demonstration mit Satz 2, der sofort aus dem Fixpunktsatz von Tychonoff und Satz 1 mit X = Y = K folgt.

Satz 2. *Es seien* E *ein lokalkonvexer, metrisierbarer topologischer Vektorraum und* K *eine nichtleere, konvexe, kompakte Teilmenge von* E. *Dann hat jeder stetiger zufälliger Operator* $F : \Omega \times K \to E$ *mit* $F(\Omega \times K) \subset K$ *einen zufälligen Fix-*

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punkt.

Die Teilmenge K des topologischen Vektorraumes heißt zulässig, wenn zu jedem Kompaktum $K_0 \subset K$ und zu jeder Nullumgebung $V \subset E$ eine finite Abbildung $f_V : K_0 \to K$ existiert mit $f_V(x) - x \in V$ $(x \in K_0)$.
Bisher ist kein Beispiel für eine nicht zulässige konvexe Menge bekannt. Jede konvexe Menge eines lokalkonvexen Raumes ist bekanntlich zulässig. Untersuchungen über die Zulässigkeit von Mengen wurden u.a. in den Arbeiten [6],[10], [11] durchgeführt. Ein topologischer Raum X heißt zusammenziehbar, wenn ein $x_0 \in X$ und eine stetige Abbildung $h : X \times [0,1] \to X$ existiert, so daß $h(x,0) = x$, $h(x,1) = x_0$ $(x \in X)$ gilt. Jede bezüglich eines $x_0 \in E$ sternförmige Teilmenge X eines topologischen Vektorraumes E ist zum Beispiel zusammenziehbar.
Aus Satz 1 und einem Resultat von Jerofsky [10] erhalten wir nun ein später verwendetes Ergebnis:

Satz 3. *Sei E ein vollständiger separabler metrischer Vektorraum, K eine nichtleere, zulässige, zusammenziehbare Teilmenge von E, die eine endliche Vereinigung von abgeschlossenen, konvexen Teilmengen von E ist. Weiter sei* $F : \Omega \times K \to K$ *ein kompekter zufälliger Operator. Dann hat F einen zufälligen Fixpunkt.*

Beweis: Wir setzen in Satz 1 X = Y = K. Für jedes $w \in \Omega$ gilt $\{x \in K : F(w,x) = x\} \neq \emptyset$ nach einem Satz von Jerofsky ([10], Satz 1.5.6, dieser Satz gilt sogar für nichtmetrisierbare topologische Vektorräume). Satz 1 liefert die Behauptung.

Wir bemerken, daß in jedem lokalkonvexen Raum die endliche Vereinigung von abgeschlossenen konvexen Mengen eine zulässige Menge ist ([10], Folgerung 1.5.4), in nichtlokalkonvexen Folgenraum ℓ^p $(0 < p < 1)$ ist zum Beispiel jede kompakte Menge K, die endliche Vereinigung von abgeschlossenen, konvexen

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Mengen ist, zulässig. (Folgerung 1.5.5. aus [1]).
Das im Beweis von Satz 3 verwendete Resultat ist das einzige deterministische Hilfsmittel, das wir in der weiteren Arbeit verwenden. Aus Satz 3 leiten wir in Abschnitt 4 einige Fixpunktaussagen ohne Rückgriff auf entsprechende deterministische Ergebnisse her. Dazu benötigen wir einen allgemeinen Homotopieerweiterungssatz.

3. HOMOTOPIEERWEITERUNGSSATZ FÜR ZUFÄLLIGE KOMPAKTE OPERATOREN

Definition. *Sei* E *ein metrischer Raum,* $X \subseteq Y \subseteq E$ *und* $F : \Omega \times Y \to E$, $G : \Omega \times X \to E$ *kompakte zufällige Operatoren. Wir nennen* F *und* G *homotop auf* X *(in Zeichen* $F \overset{X}{\sim} G$*), wenn eine Abbildung* $H : \Omega \times Y \times [0,1] \to E$ *existiert, für die gilt:*

(1) H *ist meßbar bezüglich* w *für jedes* $(x,t) \in Y \times [0,1]$

(2) H *ist kompakt bezüglich* $(x,t) \in Y \times [0,1]$ *für jedes feste* $w \in \Omega$

(3) $H(w,x,0) = F(w,x)$, $H(w,x,1) = G(w,x)$ $(x \in Y, w \in \Omega)$

(4) $x \neq H(w,x,t)$ $(w \in \Omega, x \in X, t \in [0,1])$.

H *heißt dann Homotopie zwischen* F *und* G.

Ausgehend vom Konzept von Granas [5] bewies der Verfasser in [7] einen allgemeinen Homotopieerweiterungssatz für kompakte mengenwertige Operatoren. Für den punktwertigen Fall beweisen wir nun diesen Satz für die Situation der zufälligen Operatoren. Der Satz aus [7] ist damit (für punktwertige Abbildungen und Metrisierbarkeit) ein echter Spezialfall des folgenden Satzes.

Satz 4. *Es seien* E *ein vollständiger, separabler metrischer Raum,* X *und* Y *abgeschlossene Teilmengen von* E *mit* $X \subseteq Y$ *sowie* $F : \Omega \times Y \to E$, $G : \Omega \times Y \to E$ *kompakte zufällige Operatoren.* F *sei auf* X *zu* G *homotop.* H *bezeichne eine*

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Homotopie zwischen F *und* G. *Hat* F *keinen zufälligen Fixpunkt, so existiert ein kompakter zufälliger Operator* G_1: $: \Omega \times Y \to E$ *ohne zufälligen Fixpunkt, für den* $G_1(w,x) = G(w,x)$ $(w \in \Omega, x \in X)$ *sowie* $G_1(\Omega \times Y) \subseteq H(\Omega \times Y \times [0,1])$ *gilt.*

Beweis: Wir nehmen an, jeder kompakte zufällige Operator $G_1 : \Omega \times Y \to E$ mit $G_1(w,x) = G(w,x)$ $(w \in \Omega, x \in X)$ und $G_1(\Omega \times Y) \subseteq H(\Omega \times Y \times [0,1])$ habe einen zufälligen Fixpunkt. Dann hat insbesondere auch G selbst einen zufälligen Fixpunkt, so daß alle $w \in \Omega$ stets $\{x \in Y : x = G(w,x)\} \neq \emptyset$ gilt. Wegen $G(w,x) = H(w,x,1)$ $(x \in Y, w \in \Omega)$ sind dann auch alle Mengen.

$$A(w) := \{x \in Y : x = H(w,x,t) \text{ für ein } t \in [0,1]\} \qquad (w \in \Omega)$$

nichtleer. Aus der Stetigkeit von H bezüglich (x,t) und der Kompaktheit von [0,1] folgt ferner leicht die Abgeschlossenheit jeder Menge A(w) $(w \in \Omega)$. Wir zeigen nun die schwache Meßbarkeit der Abbildung $A : \Omega \to Cl(Y)$.
Sei M eine beliebige abgeschlossene Teilmenge von Y. Es existieren abzählbar dichte Teilmengen M_o von M und I_o von [0,1]. Dann gilt

$$A^{-1}(M) = \bigcap_{n=1}^{\infty} \bigcup_{x_n \in M_o} \bigcup_{t_n \in I_o} \left\{w \in \Omega : d(x_n, H(w,x_n,t_n)) < \frac{1}{2}\right\}.$$

Ist nämlich $w \in A^{-1}(M)$, so existiert ein $x \in M$ mit $x = H(w,x,t)$ für ein $t \in [0,1]$. Für jedes $n \in N$ existiert ein $x_n \in M_o$ und ein $t_n \in I_o$ mit $d(x_n,x) < 1/2n$ und $d(H(w,x_n,t_n), H(w,x,t)) < 1/2n$. Hieraus folgt $d(x_n, H(w,x_n,t_n)) < 1/n$.
Sei umgekehrt $w \in \Omega$ so gegeben, daß für jedes $n \in N$ ein $x_n \in M_o$ und ein $t_n \in I_o$ existiert mit $d(x_n, H(w,x_n,t_n)) < 1/n$. Weil H bezüglich (x,t) kompakt ist, können wir o.B.d.A. die Konvergenz $H(w,x_n,t_n) \to y_o$ annehmen. Hieraus folgt auch $x_n \to y_o$ und $y_o \in M$. Ferner gilt o.B.d.A. die Konvergenz $t_n \to t_o \in [0,1]$ und damit $H(w,x_n,t_n) \to H(w,y_o,t_o)$, woraus

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$y_0 = H(w,y_0,t_0)$ folgt. Daher gilt in der Tat $w \in A^{-1}(M)$. Aus der Darstellung für $A^{-1}(M)$ folgt die Meßbarkeit dieser Menge, da H meßbar bezüglich w ist und nach [9], Theorem 3.5 dann auch die Abbildung $w \to d(x,H(w,x,t))$ $(w \in \Omega)$ für jedes feste $x \in Y$, $t \in [0,1]$ meßbar ist.
Damit haben wir die schwache Meßbarkeit von A gezeigt. Wir setzen nun (wobei d die Metrik in E bezeichne)

$$f(w,x) = \frac{d(x,A(w))}{d(x,A(w)) + d(x,X)} \quad (x \in E,\ w \in \Omega).$$

Wegen $x \neq H(w,x,t)$ $(x \in X,\ w \in \Omega,\ t \in [0,1])$ gilt $A(w) \cap X = \emptyset$ $(w \in \Omega)$. Ferner waren X und A(w) $(w \in \Omega)$ abgeschlossen. Somit ist $f : \Omega \times E \to \mathbb{R}$ eine Abbildung mit $f(w,x) = 0$ $(w \in \Omega,\ x \in A(w))$ und $f(w,x) = 1$, $(w \in \Omega,\ x \in X)$. Weiterhin ist $f(w,\cdot)$ stetig für jedes feste $w \in \Omega$. Weil $A : \Omega \to Cl(Y)$ schwach meßbar war, muß nach [9], Theorem 35 auch die Abbildung $w \to d(x,A(w))$ für jedes feste $x \in E$ meßbar sein. Damit ist $f(\cdot,x)$ meßbar für jedes feste $x \in E$.
Sei nun $G_1(w,x) = H(w,x,f(w,x))$ $(w \in \Omega,\ x \in Y)$. Dann ist $G_1 : \Omega \times Y \to E$ ein kompakter zufälliger Operator (die Meßbarkeit von G_1 bezüglich w folgt aus der Meßbarkeit von $f(\cdot,x)$ und Theorem 6.5 aus [9]). Für jedes $x \in X$ gilt

$$G_1(w,x) = H(w,x,1) = G(w,x) \quad (w \in \Omega).$$

Nach unserer Annahme muß G_1 einen zufälligen Fixpunkt besitzen, für jedes $w \in \Omega$ gilt damit $\{x \in Y : x = G_1(w,x)\} \neq \emptyset$. Sei $w \in \Omega$. Dann existiert also ein $x \in Y$ mit $x = G_1(w,x) = H(w,x,f(w,x))$. Wegen $f(w,x) \in [0,1]$ folgt $x \in A(w)$, also $f(w,x) = 0$ und damit $x = H(w,x,0) = F(w,x)$. Somit erhalten wir für jedes $w \in \Omega$ die Relation $\{x \in Y : x = F(w,x)\} \neq \emptyset$ und nach Satz 1 hat F einen zufälligen Fixpunkt. Dieser Widerspruch beweist Satz 4.

4. ANWENDUNGEN DES HOMOTOPIEERWEITERUNGSSATZES

Wir verwenden nun Satz 4 zur Herleitung dreier

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verschiedenartiger Fixpunktaussagen, ohne nochmals auf analoge deterministische Ergebnisse Bezug nehmen zu müssen.
Zunächst beweisen wir eine Existenzaussage mit Leray-Schauder-Randbedingung.

Satz 5. *Sei E vollständiger, separabler metrischer Vektorraum, W eine offene Nullumgebung von E und K eine bezüglich* $o \in E$ *sternförmige, zulässige Menge, die eine endliche Vereinigung von abgeschlossenen konvexen Teilmengen von E ist. Sei* $F : \Omega \times (\overline{W} \cap K) \to K$ *ein kompakter zufälliger Operator. Es gelte für alle* $w \in \Omega$ *stets* $F(w,x) \neq \beta x (x \in \partial W \cap K, \beta > 1)$. *Dann hat F einen zufälligen Fixpunkt.*

Beweis: Wir nehmen an, F habe keinen zufälligen Fixpunkt. Sei $G(w,x) = 0$ $(w \in \Omega, x \in \overline{W} \cap \overline{K})$ und $H(w,x,t) = (1-t)F(w,x)$, $(w \in \Omega, x \in \overline{W} \cap \overline{K})$, $(t \in [0,1])$.
Man erkennt leicht, daß F und G mittels H homotop auf $\partial W \cap K$ sind. Nach Satz 4 existiert ein kompakter zufälliger Operator $G_1 : \Omega \times (W \cap K) \to K$ mit $G_1(w,x) = 0$ für alle $w \in \Omega$ und $x \in \partial W \cap K$ ohne zufälligen Fixpunkt. Sei

$$T(w,x) = \begin{cases} G_1(w,x) & \text{für } x \in \overline{W} \cap K,\ w \in \Omega \\ 0 & \text{für } x \in K \setminus \overline{W},\ w \in \Omega. \end{cases}$$

Dann ist $T : \Omega \times K \to K$ ein kompakter zufälliger Operator, der nach Satz 3 einen zufälligen Fixpunkt $x_0 : \Omega \to K$ besitzen muß. Dann ist sogar $x_0(w) \in \overline{W} \cap K$ für alle $w \in \Omega$ (sonst wäre für ein $w_0 \in \Omega$ die Beziehung $T(w_0, x_0(w_0)) = 0 = x_0(w_0)$ und $x_0(w_0) \in K \setminus \overline{W}$ gültig, was wegen $0 \in W$ ein Widerspruch ist), also $G_1(w, x_0(w)) = x_0(w)$. Damit hätte G_1 doch einen zufälligen Fixpunkt.
Satz 5 ist daher bewiesen.

Unsere nächste Aussage gibt eine hinreichende Bedingung für die Existenz eines nichttrivialen zufälligen Fixpunktes für einen positiven zufälligen Operator an.

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Satz 6. *Es seien* E *ein vollständiger, separabler metrischer Vektorraum,* W *eine offene Nullumgebung sowie* K *ein abgeschlossener zulässiger Kegel aus* E. *Weiter sei* $G : \Omega \times (\overline{W} \cap K) \to K$ *ein zufälliger kompakter Operator. Es seien die beiden folgenden Bedingungen erfüllt:*

(1) Für alle $w \in \Omega$ *sei* $G(w,x) \neq \beta x \quad (x \in \partial W \cap K, \beta > 1)$

(2) Es existiere eine Nullumgebung $V \subset W$ *und eine meßbare Abbildung* $y : \Omega \to E$ *mit* $y(w) \in K \setminus \overline{V}$ *derart, daß für alle* $w \in \Omega$ *stets* $\beta x + (1 - \beta)y(w) \neq G(w,x)$ $(x \in \partial V \cap K, \beta \geq 1)$ *gilt.*

Dann hat G *einen zufälligen Fixpunkt* $x_0 : \Omega \to W \cap K$ *mit* $x_0(w) \neq 0$ *für jedes* $w \in \Omega$.

Beweis: Sei $F(w,x) = y(w)$ $(x \in \overline{V} \cap K, w \in \Omega)$ und $H(w,x,t) = tG(w,x) + (1 - t)F(w,x)$ $(w \in \Omega, x \in \overline{V} \cap K, t \in [0,1])$. Dann folgt aus (2) die Relation $x \neq H(w,x,t)$ $(w \in \Omega, x \in \partial V \cap K, t \in [0,1])$ und es gilt $F \overset{\partial V \cap K}{\sim} G$. F hat wegen $F(w,x) \notin \overline{V}$ $(w \in \Omega, x \in \overline{V} \cap K)$ auf $\overline{V} \cap K$ keinen zufälligen Fixpunkt. Nach Satz 4 existiert ein kompakter zufälliger Operator $G_1 : \Omega \times (\overline{V} \cap K) \to K$ ohne zufälligen Fixpunkt mit $G_1(w,x) = G(w,x)$ $(x \in \partial V \cap K, w \in \Omega)$. Dann muß aber $G|\Omega \times [(W \setminus V) \cap K]$ einen zufälligen Fixpunkt $x_0 : \Omega \to (\overline{W \setminus V}) \cap K$ besitzen, denn sonst wäre durch

$$T(w,x) = \begin{cases} G_1(w,x) & (x \in \overline{V} \cap K, \quad w \in \Omega) \\ G(w,x) & (x \in (\overline{W \setminus V}) \cap K, \; w \in \Omega) \end{cases}$$

ein kompakter zufälliger Operator $T : \Omega \times (\overline{W} \cap K) \to K$ erklärt, der keinen zufälligen Fixpunkt hat.
Jedoch gilt wegen $T(w,x) = G(w,x)$ $(x \in \partial W \cap K, w \in \Omega)$ und (1) auch $T(w,x) \neq \beta x$ $(w \in \Omega, x \in \partial W \cap K, \beta > 1)$ und T hat nach Satz 5 doch einen zufälligen Fixpunkt. Damit hat G einen zufälligen Fixpunkt $x_0 : \Omega \to E$ mit $x_0(w) \neq 0$ für alle $w \in \Omega$.
Die Voraussetzungen (1) und (2) von Satz 6 sind schwächer als die bekannten Voraussetzungen von Krasnoselski in einem entsprechenden deterministischen Resultat (s.z. B.[16], S. 154). Wir bemerken, daß für lokalkonvexes E die Zulässig-

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keit von K stets erfüllt ist und für konvexes W die Voraussetzung (1) z. B. durch die Bedingung $G(\Omega \times (\partial W \cap K)) \subseteq \overline{W} \cap K$ gesichert ist (dies gilt natürlich auch für Satz 5).

Am Schluß unserer Arbeit sei E ein separabler Banachraum, W eine offene, konvexe Teilmenge von E (mit o.B.d.A. $0 \in W$) und $F : \Omega \times \overline{W} \to E$ ein kompakter zufälliger Operator. Für die Anwendung von Näherungsverfahren ist es zweckmäßig, neben F eine Folge von kompakten, zufälligen Operatoren $F_n : \Omega \times \overline{W} \to E$ $(n = 1,2,...)$ zu betrachten, die oft finit sein werden und für $n \to \infty$ 'nahe' bei F liegen. Es interessieren Bedingungen, unter denen aus der Existenz und Eindeutigkeit eines zufälligen Fixpunktes x_0 von F die Existenz von zufälligen Fixpunkten x_n von F_n $(n \geq n_0)$ und die Konvergenz von (x_n) gegen x_0 in einem möglichst starkem Sinne folgt. Wir geben im abschließenden Satz 7 als erneute Anwendung von Satz 4 hinreichende Bedingungen dafür an, daß diese Konvergenz sogar gleichmäßig erfolgt.
Um die Eindeutigkeit des zufälligen Fixpunktes von F zu sichern, verwenden wir ein Resultat von Smith und Stuart [15] und fordern:
"Für jedes $w \in \Omega$ sei die Abbildung $F(w,\cdot)$ stetig Frechet-differenzierbar in W und die Mengen

(S) $\{x \in W : 1 \text{ ist Eigenwert von } F'(w,\cdot)(x)\}$

haben für jedes $w \in \Omega$ keinen Häufungspunkt in W."
Dann gilt:

Satz 7. *Sei* $\dim(E) > 1$ *und* Ω *zusätzlich ein kompakter topologischer Raum. Der kompakte zufällige Operator* $F : \Omega \times \overline{W} \to \overline{W}$ *sei gleichmäßig stetig bezüglich* $(w,x) \in \Omega \times \overline{W}$ *und es gelte* $F(w,x) \neq x$ $(x \in \partial W, w \in \Omega)$. *Gilt die Bedingung (S) und konvergieren die kompakten zufälligen Operatoren* $F_n : \Omega \times \overline{W} \to E$ *gleichmäßig bezüglich* $w \in \Omega$ *und* $x \in \overline{W}$ *gegen F, so haben F genau einen zufälligen Fixpunkt* $x_0 : \Omega \to \overline{W}$ *und alle* F_n $(n \geq n_0)$ *einen zufälligen Fixpunkt*

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$x_n : \Omega \to \overline{W}$ *und für jede Folge* (x_n) *von zufälligen Fixpunkten von* F_n *gilt die gleichmäßige Konvergenz*

$$\sup_{w \in \Omega} \|x_n(w) - x_0(w)\| \to 0.$$

Beweis: Nach Smith und Stuart [15] hat für jedes $w \in \Omega$ die Abbildung $F(w,\cdot)$ genau einen Fixpunkt. Aus dem Beweis von Satz 1 erkennt man sofort, daß dann F genau einen zufälligen Fixpunkt $x_0 : \Omega \to \overline{W}$ hat.

Sei $\varepsilon > 0$ beliebig. Wir zeigen, daß ein $n_0 \in \mathbb{N}$ existiert, so daß alle F_n $(n \geq n_0)$ einen zufälligen Fixpunkt $x_n : \Omega \to \overline{W}$ haben und für diese $\|x_n(w) - x_0(w)\| < \varepsilon$ für alle $w \in \Omega$ und $n \geq n_0$ gilt. Sei dazu $M(w) := \{x \in \overline{W} : \|x - x_0(w)\| \geqslant \varepsilon\}$ $(w \in \Omega)$. $M(w)$ ist für jedes $w \in \Omega$ abgeschlossen und es gilt $x \neq F(w,x)$ $(w \in \Omega, x \in M(w))$. Wegen der Kompaktheit von $F(w,\cdot)$ und der Abgeschlossenheit von $M(w)$ ist für jedes $w \in \Omega$ die Menge $\{z \in E : z = x - F(w,x), x \in M(w)\}$ abgeschlossen. Wegen $0 \notin x - F(w,x)$ $(w \in \Omega, x \in M(w))$ gilt $\rho(w) := \inf\{\|x - F(w,x)\|, x \in M(w)\} > 0$ für jedes $w \in \Omega$. Aus der gleichmäßigen Stetigkeit von F folgt leicht die Stetigkeit von $\rho : \Omega \to \mathbb{R}^+$. Da Ω kompakt ist, hat ρ ein Minimum.

Sei $\delta_0 := \min_{w \in \Omega} \rho(w)$. Weiterhin gilt nach Voraussetzung $x \neq F(w,x)$ für alle $x \in \partial W$ und alle $w \in \Omega$. Mit analogen Überlegungen wie soeben ergibt sich die Existenz eines $\delta_1 > 0$ derart, daß $\|x - F(w,x)\| > \delta_1$ $(x \in \partial W, w \in \Omega)$ gilt.

Wir setzen $\delta := \min\{\delta_0, \delta_1\}$ und wählen ein $n_0 \in \mathbb{N}$ derart, daß für alle $n \geq n_0$ stets $\|F_n(w,x) - F(w,x)\| < \delta$ $(w \in \Omega, x \in \overline{W})$ gilt.

Sei

$$H(w,x,t) := (1 - t)F(w,x) + tF_n(w,x)$$
$$(w \in \Omega, x \in \overline{W}, t \in [0,1], n \geq n_0).$$

Dann gilt $x \neq H(w,x,t)$ für alle $w \in \Omega$, $x \in \partial W$ und $t \in [0,1]$. Sonst würde ein $w' \in \Omega$, $x' \in \partial W$, $t' \in [0,1]$ existieren mit

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$$x' = (1 - t')F(w',x') + t'F_n(w',x'),$$

woraus $\|x' - F(w',x')\| < \delta$ folgen würde. Dies würde $\|x - F(w,x)\| > \delta_1 (x \in \partial W, w \in \Omega)$ widersprechen. Somit sind F und F_n $(n \geq n_o)$ homotop auf $\overline{W}$. Dann hat F_n $(n \geq n_o)$ auf $\overline{W}$ einen zufälligen Fixpunkt, denn sonst würde nach Satz 4 ein kompakter zufälliger Operator $G_1 : \Omega \times \overline{W} \to E$ ohne zufälligen Fixpunkt existieren, für den $G_1(w,x) = F(w,x)$ $(w \in \Omega, x \in \partial W)$ gilt. Wegen $F(w,x) \in \overline{W}$ $(w \in \Omega, x \in \overline{W})$ gilt $G_1(\Omega \times \partial W) \subseteq \overline{W}$ und das impliziert mit der Konvexität von W die Bedingung $G_1(w,x) \neq \beta x$ $(\beta > 1, x \in \partial W)$ für alle $w \in \Omega$. Dann müßte aber nach Satz 3 (mit K = E) G_1 doch einen zufälligen Fixpunkt haben.
Sei nun für jedes $n \geq n_o$ ein zufälliger Fixpunkt $x_n : \Omega \to \overline{W}$ von F_n gewählt.

Angenomen, es gilt für ein $w_o \in \Omega$ und ein $n \geq n_o$ die Beziehung $\|x_n(w_o) - x_o(w_o)\| \geq \varepsilon$. Dann ist $\overline{x} := x_n(w_o) \in M(w_o)$ und es gilt daher $\|\overline{x} - F(w_o,\overline{x})\| \geq \rho(w_o) > \delta_o \geq \delta$. Wegen $\overline{x} = F_n(w_o,\overline{x})$ gilt andererseits $\|\overline{x} - F(w_o,\overline{x})\| < \delta$. Dieser Widerspruch zeigt die Gültigkeit von $\|x_n(w) - x_o(w)\| < \varepsilon$ für alle $w \in \Omega$ und alle $n \geq n_o$ und die gleichmäßige Konvergenz von (x_n) gegen x_o ist gezeigt.

Bemerkung: Statt (S) kann natürlich gefordert werden, daß F höchstens einen zufälligen Fixpunkt hat. Aus der Bemerkung nach Satz 1 folgt dann, daß jedes $F(w,\cdot)$ genau einen Fixpunkt besitzt $(w \in \Omega)$. (Die Existenz ist wegen $F(\Omega \times \overline{W}) \subseteq \overline{W}$ gesichert).

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REZIME

O STOHASTIČKIM NEPOKRETNIM TAČKAMA NEPREKIDNIH STOHASTIČKIH OPERATORA U METRIČKIM VEKTORSKIM PROSTORIMA

U ovom radu je dokazan stav o homotopijskom proširenju za stohastičke operatore. On je uopštenje odgovarajućeg determinističkog rezultata iz [1] i omogućava izvodjenje raznih teorema o nepokretnoj tački za stohastičke operatore bez korišćenja odgovarajućih determinističkih rezultata.

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THE QUASIASYMPTOTIC OF DISTRIBUTIONS AND THE DISTRIBUTIONAL STIELTJES TRANSFORMATION

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ABSTRACT

Using the results for the quasiasymptotic at $\pm\infty$ of distributions ([8],[9]), the paper obtains a final value Abelian theorem and a Tauberian theorem of the Keldyš type for the distributional Stieltjes transformation.

1. NOTIONS AND NOTATION

The quasiasymptotic at ∞ of tempered distributions which have their supports in $[0,\infty)$ was studied by the Soviet mathematicians Vladimirov, Drožinov and Zavialov (see [12] and the references given there). Using this notion they obtained remarkable results. S.Pilipović [8] extended this notion to the space of Schwartz distributions (on the real line):

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DEFINITION 1. *([8]) It is said that an* $f \in \mathcal{D}'$ *has the quasiasymptotic at* $\pm\infty$ *with respect to some positive continuous function* $c(k)$, $k \in (a, +\infty)$, $a > 0$ *if for some* $g \in \mathcal{D}'$, $g \neq 0$,

$$\lim_{k\to\infty} \left\langle \frac{f(kx)}{c(k)}, \phi(x) \right\rangle = \langle g(x), \phi(x) \rangle , \quad \phi \in \mathcal{D} . \tag{1.0}$$

In this case we write $f \overset{q}{\sim} g$ *at* $\pm\infty$ *with respect to* $c(k)$.

THEOREM I *([8]) Let* $f \in \mathcal{D}'$ *have the quasiasymptotic at* $\pm\infty$ *with respect to some positive continuous function* $c(k), k > 0$. *Then*

(i) $f \in S'$;

(ii) There are $\nu \in \mathbb{R}$ *and a slowly varying function* $L(k)$, $k > a$, *such that*

$$c(k) = k^{\nu} L(k), \quad k > a .$$

Moreover, g *is a homogeneous distribution with the order of homogenity* ν ;

(iii) If $\nu > -1$, *then (1.0) holds in the sense of convergence in* S' *(for* $\phi \in S$) ;

(iv) If $\nu = -1$ *and* $\frac{1}{L(k)}$, $k > a$, *is bounded, then the assertion in (iii) holds, as well.*

COROLLARY II *([8]) Let* $f \in \mathcal{D}'$ *and* $f \overset{q}{\sim} g$ *at* $\pm\infty$ *with respect to* $k^{\nu} L(k)$, *where* $\nu \in \mathbb{R} \setminus (-\mathbb{N})$. *Then (1.0) holds in the sense of convergence in* S'.

Let us recall that the scale of homogeneous distribution $f_{\nu+1}$, $\nu \in \mathbb{R}$, is defined in the following way ([11]):

$$f_{\nu+1}(x) = \begin{cases} \dfrac{H(x)x^{\nu}}{\Gamma(\nu+1)} & \text{for } \nu > -1 \\ f^{(n)}_{\nu+n+1}(x) & \text{for } \nu \leqslant -1,\ n+\nu > -1,\ n \in \mathbb{N}, \end{cases} \qquad (x \in \mathbb{R})$$

where H is the Heaviside function.

(We also use the notation $H(x)x^{\nu} = x_+^{\nu}$, $H(-x)|x|^{\nu} = x_-^{\nu}$, $\nu > -1$).

The following two theorems are needed for our investigations.

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THEOREM A. *([9]) (i) Let* F *be a locally integrable function and* $\nu \in \mathbb{R}$, $\nu > -1$, *such that*

$$\lim_{\substack{x\to+\infty \\ x\to-\infty}} \frac{F(x)}{|x|^{\nu}L(|x|)} = C_{\pm} \quad \text{where } (C_+,C_-) \neq (0,0) .$$

Then $F \overset{q}{\sim} g$ *at* $\pm\infty$ *with respect to* $k^{\nu}L(k)$ *where*

$$g(x) = \bar{C}_+ f_{\nu+1}(x) + \bar{C}_- f_{\nu+1}(-x), \quad x \in \mathbb{R},$$

$$(\bar{C}_+,\bar{C}_-) \neq (0,0) .$$

(ii) If $f \overset{q}{\sim} g$ *at* $\pm\infty$ *with respect to* $k^{\nu}L(k)$ *then* $f^{(m)} \overset{q}{\sim} g^{(m)}$ *at* $\pm\infty$ *with respect to* $k^{\nu-m}L(k)$.

(iii) Let $f \in \mathcal{D}'$ *and* $f \overset{q}{\sim} g$ *at* $\pm\infty$ *with respect to* $k^{\nu}L(k)$ *where* $\nu \in \mathbb{R} \setminus (-\mathbb{N})$. *There are* $m \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$ *and a continuous function* F *such that* $m+\nu > -1$,

$$f = F^{(m)} \text{ and } \lim_{\substack{x\to+\infty \\ x\to-\infty}} \frac{F(x)}{|x|^{\nu+m}L(|x|)} = C_{\pm} ,$$

where $(C_+,C_-) \neq (0,0)$.

Let T be the class of non-decreasing functions defined for $x \geqslant 0$ and identically equal to zero in a neighbourhood of the origin. For $\alpha,\beta > 0$ we denote by $T(\alpha,\beta)$ the set of all $\phi \in T$ which are differentiable and satisfy the inequalities

$$\alpha\phi(x) < x\phi'(x) < \beta\phi(x)$$

for $x > a$, $a = \text{const.} > 0$.

THEOREM B *([10]) Let* $\phi(x) \in T(\alpha',\beta')$ *and* $\psi(x) \in T$ *for* $x \geqslant 0$. *Suppose also that either* $\phi(-x) \in T(\alpha',\beta')$, $\psi(-x) \in T$ *or* $-\phi(-x) \in T(\alpha',\beta')$, $-\psi(-x) \in T$ *when* $x \leqslant 0$. *Assume that*

$$A_1 \leqslant \left|\frac{\phi(-x)}{\phi(x)}\right| \leqslant A_2$$

for large values of x, *where* A_1 *and* A_2 *are positive constants.*

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Suppose that the relation $m < \alpha' < \beta'$ *holds for* $m = [\beta']$. *If*

$$\int_{-\infty}^{+\infty} (u-z)^{-m-1} d\phi(u) \sim \int_{-\infty}^{+\infty} (u-z)^{-m-1} d\psi(u)$$

when $z \to \infty$ *along a non-real half-ray from the origin, then* $\phi(x) \sim \psi(x)$ *when* $x \to +\infty$ *and* $x \to -\infty$.

We shall follow the definition of the distributional Stieltjes transformation given in [6], [7].

DEFINITION 2. *Let* $f \in S'$. *We say that* $f \in J'(r)$, *if there exist* $m \in \mathbb{N}_0$ *and a locally integrable function* F *such that*

a) $f = F^{(m)}$, supp $F \subset [0,\infty)$;

(1.1) *b)* $\int_{-\infty}^{+\infty} |F(x)(x+z)^{-r-m-1}| dx < \infty$ *for* $\operatorname{Im} z \neq 0$:

The Stieltjes transformation S_r of index r, $r \in \mathbb{R} \setminus (-\mathbb{N}_0)$, of a distribution $f \in J'(r)$ with the properties given in (1.1) is a complex valued function given by

$$(1.2) \qquad (S_r f)(z) = (r+1)_m \int_{-\infty}^{+\infty} F(x)(x+z)^{-r-m-1} dx$$

$$= (r+1)_m \left\langle F(x), \frac{1}{(x+z)^{r+m+1}} \right\rangle$$

for $\operatorname{Im} z \neq 0$, where $(r)_k = r(r+1)\ldots(r+k-1)$, $k > 0$ and $(r)_0 = 1$.

It is easy to see that $(S_r f)(z)$ is a holomorphic function of the complex variable z in the domain $\mathbb{C} \setminus (-\infty,\infty)$.

We shall observe in this paper the function $(S_r f)$ in the upper half plane i.e. when $\operatorname{Im} z > 0$. This is not a restriction because one can easily show that all the assertion which are to follow hold with $\operatorname{Im} z < 0$.

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2. ABELIAN AND TAUBERIAN THEOREM

First we shall prove the following lemma which will be used in the proof of the Abelian theorem. Assume that assumptions of Theorem A part (iii) hold for f with $L \equiv 1$. Moreover, assume that $r \in \mathbb{R} \setminus (-\mathbb{N})$ and $\nu < r$.

LEMMA. *The function*

$$z \mapsto z^{r-\nu}(S_r f)(i+z) \ ,$$

is uniformly bounded in the domain

$$\Lambda_\varepsilon = \{Re^{i\phi} : R > 0,\ \varepsilon \leqslant \phi \leqslant \pi-\varepsilon\},\ 0 < \varepsilon < \frac{\pi}{2} \ .$$

Proof. Since

$$R^2-2R|t|\cos\varepsilon + t^2 + 1 \geqslant (R^2+t^2)(1+\cos\varepsilon)-(R+|t|)^2\cos\varepsilon + 1$$

$$\geqslant (R^2+t^2)(1+\cos\varepsilon)-2(R^2+t^2)\cos\varepsilon + 1$$

$$\geqslant (R^2+t^2)(1-\cos\varepsilon) + 1$$

we have, for $z \in \Lambda_\varepsilon$

$$|i+z+t| = (R^2+t^2+2R\,t\cos\phi + 2R\sin\phi+1)^{1/2}$$

$$\geqslant (R^2-2R|t|\cos\varepsilon+t^2+1)^{1/2}$$

$$\geqslant \sqrt{(R^2+t^2+1)(1-\cos\varepsilon)} \geqslant (R+|t|+1)\sqrt{\frac{1-\cos\varepsilon}{3}} \ .$$

Using the fact that for some $C > 0$

$$|F(x)| \leqslant C(1+|x|^{\nu+m}) \qquad , x \in \mathbb{R}$$

we have (with suitable C_1 and C_2)

$$|(S_r f)(i+z)| \leqslant (r+1)_m \int_{-\infty}^{+\infty} |F(x)(z+x+i)^{-r-m-1}|dx$$

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$$\leqslant (r+1)_m C\Big(\frac{3}{1-\cos\varepsilon}\Big)^{(r+m+1)/2} \int_{-\infty}^{+\infty} \frac{1+|x|^{\nu+m}}{(R+|x|+1)^{r+m+1}}\,dx$$

$$= C_1\Big(\int_0^{\infty} (R+x+1)^{-r-m-1}dx + \int_0^{\infty} x^{\nu+m}\,(R+x+1)^{-r-m-1}dx\Big)$$

$$= C_1\Big[\frac{1}{(r+m)(R+1)^{r+m}} + \frac{\Gamma(r-\nu)\Gamma(\nu+m+1)}{\Gamma(r+m+1)}\,\frac{1}{(R+1)^{r-\nu}}\Big]$$

$$\leqslant C_2(R+1)^{\nu-r} \quad \text{(see [5, p. 233])}.$$

This completes the proof of the lemma.

ABELIAN THEOREM. *(i) Let* $f \in S'$ *and* $f \overset{q}{\sim} g$ *at* $\pm\infty$ *with respect to* $k^{\nu}L(k)$, $\nu \in \mathbb{R}\setminus(-\mathbb{N})$. *Let* $r \in \mathbb{R}\setminus(-\mathbb{N})$ *and* $r > \nu$. *Then, there is* $(C_+,C_-)\neq(0,0)$ *such that for any* $z \in \mathbb{C}\setminus\mathbb{R}$

$$\frac{(S_r f)(kz)}{k^{\nu-r}L(k)} \to \frac{\Gamma(r-\nu)}{\Gamma(r+1)}\, z^{\nu-r}\,(C_+ + C_- e^{-\pi i(\nu-2r-m-1)}) \quad \text{as } k \to \infty$$

(ii) If $L \equiv 1$ *this holds uniformly in any angle of the form*

$$\Lambda_\varepsilon = \{Re^{i\phi},\ R > 0,\ \varepsilon \leqslant \phi \leqslant \pi-\varepsilon\} \quad (0 < \varepsilon < \tfrac{\pi}{2}).$$

Proof. First, we shall prove (i) for $r > \nu+1$.

According to Theorem A (iii) we have that $f = F^{(m)}$ for some $m \in \mathbb{N}_0$ and some continuous function F such that $m+\nu > -1$ and that for every $\phi \in S$

$$\lim_{k\to\infty} \Big\langle \frac{F(kx)}{k^{m+\nu}L(k)}, \phi(x)\Big\rangle = \langle C_+ f_{\nu+m+1}(x) + C_- f_{\nu+m+1}(-x), \phi(x)\rangle, \tag{2.1}$$

where $(C_+,C_-) \neq (0,0)$.

Moreover, we have that $f \in J'(r)$ for $r > \nu$ because

$$F(x) \sim \frac{C_+ x_+^{\nu+m}L(x)}{\Gamma(\nu+m+1)}, \quad x \to \infty$$

$$F(x) \sim \frac{C_- x_-^{\nu+m}L(|x|)}{\Gamma(\nu+m+1)}, \quad x \to -\infty.$$

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If C_+ or C_- is equal to 0, then we have to substitute the symbol $\sim$ with the small "o".

The behaviour of F implies that (2.1) holds in the sense of convergence in S'_p for $p > \nu+m+1$ (see [11]).

Since the function $x \mapsto (x+z)^{-r-m-1}$ belongs to S_{r+m} $(\mathrm{Im}\, z > 0)$, the assumption $\nu < r-1$ implies that for $p = r+m$, (2.1) holds with $\phi(x) = (x+z)^{-r-m-1}$, $x \in \mathbb{R}$. This implies

$$\lim_{k\to\infty} \frac{(S_r f)(kz)}{k^{\nu-r}L(k)} = (r+1)_m \lim_{k\to\infty} \left\langle \frac{F(x)}{k^{\nu-r}L(k)}, \frac{1}{(kz+x)^{r+m+1}} \right\rangle$$

$$= \left\langle (r+1)_m \lim_{k\to\infty} \left\langle \frac{F(kx)}{k^{\nu+m}L(k)}, \frac{1}{(z+x)^{r+m+1}} \right\rangle \right.$$

$$= (r+1)_m \left(\left\langle \frac{C_+}{\Gamma(\nu+m+1)} x_+^{\nu+m}, (z+x)^{-r-m-1} \right\rangle \right.$$

$$\left. + \left\langle \frac{C_-}{\Gamma(\nu+m+1)} x_-^{\nu+m}, (x+z)^{-r-m-1} \right\rangle \right)$$

$$= (r+1)_m \left(\frac{C_+\Gamma(\nu+m+1)\Gamma(r-\nu)}{\Gamma(\nu+m+1)\Gamma(r+m+1)} z^{\nu-r} \right.$$

$$\left. + \frac{C_-\Gamma(\nu+m+1)\Gamma(r-\nu)\,(-z)^{\nu-r}}{\Gamma(\nu+m+1)\Gamma(r+m+1)(-1)^{r+m+1}} \right) \quad ([5, \text{ p. } 233]).$$

Thus we have

$$\lim_{k\to\infty} \frac{(S_r f)(kz)}{k^{\nu-r}L(k)} = A(z),$$

where

$$(2.2)\quad A(z) = \frac{C_+\Gamma(r-\nu)}{\Gamma(r+1)} z^{\nu-r} + \frac{C_-\Gamma(r-\nu)}{\Gamma(r+1)} e^{-\pi i(\nu-2r-m-1)} z^{\nu-r}$$

Now we shall prove (i) in case $r > \nu \geq r-1$.

Lebesgue's theorem implies

$$(S_r f)(kz) = (r+1)_m \left\langle F(x), \frac{1}{(kz+x)^{r+m+1}} \right\rangle \to 0 \text{ as } k \to \infty$$

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Because of

$$\frac{d}{dk}\left((S_r f)(kz)\right) = -z(r+1)(S_{r+1} f)(kz)$$

we have $(k > 0,\ \mathrm{Im}\, z > 0)$

$$(S_r f)(kz) = -\int_k^\infty \left((S_r f)(tz)\right)' dt = z(r+1)\int_k^\infty (S_{r+1} f)(tz)\,dt.$$

From the proof of the first part of the Abelian theorem, it follows that

$$\frac{(S_{r+1} f)(kz)}{k^{\nu-(r+1)} L(k)} \to (C_+ + C_-(e^{-\pi i})^{\nu-2r-m-3})\,\frac{\Gamma(r+1-\nu)}{\Gamma(r+2)}\, z^{\nu-r-1}$$

as $k \to \infty$.

Also, as $k \to \infty$

$$(S_r f)(kz) \sim z(r+1)\,\frac{\Gamma(r+1-\nu)}{\Gamma(r+2)}\, z^{\nu-r-1}$$

$$\cdot\,(C_+ + C_-(e^{-\pi i})^{\nu-2r-m-3}) \int_k^\infty t^{\nu-r-1} L(t)\,dt\,.$$

Since (see [1])

$$\int_k^\infty t^{\nu-r-1} L(t)\,dt \sim k^{\nu-r} L(k) \int_1^\infty u^{\nu-r-1}\,du = \frac{k^{\nu-r} L(k)}{r-\nu}\,, \quad k \to \infty,$$

we obtain

$$(S_r f)(kz) \sim k^{\nu-r} L(k) A(z),\ k \to \infty.$$

where $A(z)$ is given by (2.2).

Proof of (ii). Let $L \equiv 1$. Since we have proved in (i) that

$$(2.3)\quad (S_r f)(ki) \sim \frac{\Gamma(r-\nu)}{\Gamma(\nu+1)}\,(ki)^{\nu-r}(C_+ + C_-(e^{-\pi i})^{\nu-2r-m-1})$$

as $k \to \infty$,

we have

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$$(2.4)\quad (S_r f)(ki+i) \sim \frac{\Gamma(r-\nu)}{\Gamma(\nu+1)}(ki)^{\nu-r}(C_+ + C_-(e^{-\pi i})^{\nu-2r-m-1})$$

as $k \to \infty$

The Lemma implies that for $z \to (S_r f)(z+i)$, $z \in \Lambda_\varepsilon$ the conditions of Montel's Theorem ([2], p.5) are satisfied. So, we have that (2.4) (with z instead of ki) holds uniformly in Λ_ε.

Obviously, this implies that (2.3) holds (with z instead ki) uniformly in $\Lambda_{\varepsilon'}$ for any $\varepsilon' < \varepsilon$.

This completes the proof of (ii).

Let F and F_1 be continuous functions such that $\operatorname{supp}(F-F_1)$ is a compact set and

$$\lim_{\substack{x\to+\infty \\ x\to-\infty}} \frac{F(x)}{|x|^{\nu}L(|x|)} = \lim_{\substack{x\to+\infty \\ x\to-\infty}} \frac{F_1(x)}{|x|^{\nu}L(|x|)} = C_{\pm}$$

where $(C_+, C_-) \neq (0,0)$.

Then for $r > \nu$ and any z (with $\operatorname{Im} z > 0$) there holds

$$\lim_{k\to\infty} \frac{(S_r(F-F_1))(kz)}{k^{\nu-r}L(k)} = 0.$$

This means

$$(S_r F)(kz) \sim (S_r F_1)(kz) \quad \text{as } k \to \infty,\ \operatorname{Im} z > 0.$$

This fact implies that in Theorem B we can assume that the function ϕ is differentiable for $|x| > a > 0$.

TAUBERIAN THEOREM. *Let* $\nu = r-1+2\varepsilon$, $0 < \varepsilon < 1/3$.

(i) Let ψ *be a non-decreasing function such that*

$$\int_{-\infty}^{+\infty} (x+nz)^{-r-k} d\psi(x) = (r+k)\int_{-\infty}^{+\infty} (x+nz)^{-r-k-1}\psi(x)dx$$

$$\sim n^{\nu-r}L(n)z^{\nu-r}\Gamma(\nu+k+1)\frac{\Gamma(r-\nu)}{\Gamma(r+k)}(C_1 - C_2 e^{-\pi i(\nu-2r-k-1)})$$

as $n \to \infty$ *where* $C_1, C_2 > 0$.

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Then

$$\lim_{\substack{x\to+\infty\\ x\to-\infty}} \frac{\psi(x)}{|x|^{\nu+k}L(|x|)} = \begin{cases} C_1 \\ -C_2 \end{cases}$$

(ii) Let $f = \psi^{(k)}$, $k+\nu > -1$, *where* ψ *is a non-decreasing function. If* $\frac{(S_r f)(nz)}{(r+1)_{k-1}} \sim \frac{n^{\nu-r}L(n)\Gamma(\nu+k+1)\,\Gamma(r-\nu)}{\Gamma(r+k)}\, z^{\nu-r}(C_1 - C_2 e^{-\pi i(\nu-2r-k-1)})$, $n\to\infty$ $(C_1, C_2 > 0)$, *then* f *has the quasi-asymptotic at* $\pm\infty$ *with respect to* $n^{\nu}L(n)$ *and with the limit* $(C_1 x_+^{\nu+k} - C_2 x_-^{\nu+k})^{(k)}$.

Proof. (i) Let us put $\alpha' = r+k-1+\varepsilon$, $\beta' = r+k-1+3\varepsilon$ and

$$\phi(x) = \begin{cases} C_1 x^{\nu+k}L(x), & x > a \\ 0, & -a \le x \le a \\ -C_2|x|^{\nu+k}L(|x|), & x < -a \end{cases}$$

Then functions ϕ and ψ satisfy the conditions of Theorem B. By the Abelian theorem we have

$$\int_{-\infty}^{+\infty} (x+nz)^{-r-k}\, d\phi(x) = (r+k)\int_{-\infty}^{+\infty} (x+nz)^{-r-k-1}\phi(x)\,dx$$

$$\sim n^{\nu-r}L(n)z^{\nu-r}\Gamma(\nu+k+1)\,\frac{\Gamma(r-\nu)}{\Gamma(r+k)}(C_1 - C_2 e^{-\pi i(\nu-2r-k-1)}) \text{ as } n\to\infty.$$

So, Theorem B implies the assertion.

(ii) We have

$$\frac{(S_r f)(nz)}{(r+1)_{k-1}} = (r+k)\int_{-\infty}^{+\infty} (x+nz)^{-r-k-1}\psi(x)\,dx =$$

$$= \int_{-\infty}^{+\infty} (x+nz)^{-r-k}\, d\psi(x)$$

$$\sim n^{\nu-r}L(n)\Gamma(\nu+k+1)\,\frac{\Gamma(r-\nu)}{\Gamma(r+k)}\, z^{\nu-r}(C_1 - C_2 e^{-\pi i(\nu-2r-k-1)})$$

as $n\to\infty$.

From the first part of this theorem and Theorem A (i) it follows

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$\psi \overset{q}{\sim} C_1 x_+^{\nu+k} - C_2 x_-^{\nu+k}$ with respect to $n^{\nu+k} L(n)$, $n \to \infty$.

Now Theorem A (ii) implies the assertion.

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REZIME

KVASIASIMPTOTIKA DISTRIBUCIJA I DISTRIBUCIONA STIELTJESOVA TRANSFORMACIJA

Koristeći rezultate [8],[9] za kvaziasimptotiku u $\pm\infty$ temperiranih distribucija dobivene su dve teoreme jedna Abelova a druga Tauberova tipa Keldiša za distribucionu Stieltjesovu transformaciju.

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PERTURBATIONS OF A LINEAR DIFFERENCE EQUATION

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ABSTRACT

The paper deals with an n-th order linear difference equation and gives conditions under which its linear perturbation preserves the same l^p- affiliation.

1. INTRODUCTION

In the theory of ordinary differential equations there is a well known

Theorem (Weyl [1,2]). *Consider the equation*

$$y'' + p(t)y = 0, \quad t \geq 0, \tag{1}$$

along with the perturbed linear equation

$$y'' + (p(t) + q(t))y = 0, \quad t \geq 0, \tag{2}$$

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where both p(t) *and* q(t) *are locally integrable on* $[0,\infty)$.

Then, the following Weyl alternative holds: if all solutions of (1) are (not) in $L^2[0,\infty)$ *and* $q(t) \in L^2[0,\infty)$ *then all solutions of (2) are (not) in* $L^2[0,\infty)$.

Weyl's alternative result was extended by Patula and Wong [3], to include arbitrary $L^p[0,\infty)$-perturbation q(t). Also, Mironov [4] extended this theorem to the case of the delay-differential equation, while Wyrwinska [5] considered some more general classes of perturbations. Wong [6] obtained related results for a restrictive class of 2n-th order differential equations. All these results were generalized and improved in paper [7] to the general n-th order linear differential equation and corresponding linear and non-linear perturbations with or without retardations.

Recently, Butler and Rao [8, Theorem 4.1.] improved the result of Patula and Wong. Using a technique similar to this in paper [7], we are able to generalize and modify this result to a case of the general n-th order linear difference equation

$$(3) \qquad Dy \equiv y(n+m) + p_{n-1}(m)y(n+m-1) + \ldots + p_0(m)y(m) = 0, \quad m = 0,1,\ldots$$

and its linear perturbation of the form

$$(4) \qquad D_L y \equiv Dy + q_{n-1}(m)y(n+m-1) + \ldots + q_0(m)y(m) = 0, \quad m = 0,1,\ldots .$$

2. PRELIMINARIES

In this section we shall introduce some notation and list some well known results which will be used in the sequel.

Our first result is a discrete form of the variation of the parameters formula. The symbol Δ is a forward difference operator i.e. $\Delta y(m) = y(m+1) - y(m)$.

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Lemma 1. *Suppose* $F(u,v)$ *is a continuous function on* R^2 *and* y *is the solution of* $Dy = F(m,y(m))$. *Then*

$$\Delta^j y(m) = \sum_{i=1}^{n} c_i \Delta^j y_i(m) + \sum_{s=0}^{m-1} \Delta_m^j G(s,m) F(s,y(s)),$$

$$j = 0,\ldots,n-1,$$

where

$$G(s,m) = \frac{\begin{vmatrix} y_1(s+1) & \ldots & y_n(s+1) \\ \cdot & \ldots & \cdot \\ \cdot & \ldots & \cdot \\ \cdot & \ldots & \cdot \\ y_1(s+n-1) & \ldots & y_n(s+n-1) \\ y_1(m) & & y_n(m) \end{vmatrix}}{\begin{vmatrix} y_1(s+1) & \ldots & y_n(s+1) \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & \cdot \\ y_1(s+n) & & y_n(s+n) \end{vmatrix}} = \frac{W(s,m)}{D(s+1)}.$$

$D(s)$ is Casorati's determinant and for it we have

Lemma 2. (Heymann's theorem [9, p. 354-358]). *Casorati's determinant* $D(s)$ *satisfies the linear difference equation of the first order*

$$D(s+1) = (-1)^n p_0(s) D(s).$$

We have also need the following, obvious

Lemma 3. *If the sequence* $u \in \ell^p$ $(p \geq 1)$, *then it is also true for* Δu.

The next result is the discrete generalization of the fundamental inequality of Gronwall and Bellman as far as some non-linear variants of that inequality.

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Lemma 4. (Willet and Wong [10]). *Suppose that* $v(n)$, $w(n)$ *and* $u(n+1)$, $(n = 0,1,\ldots)$ *are non-negative sequences of numbers with* $v(0) = w(0) = 0$ *and that* u_0 *and* p *are constants such that* $u_0 > 0$ *and* $p \geq 0$, $p \neq 1$. *Then the inequality*

$$u(n+1) \leq u_0 + \sum_{j=0}^{n} v(j)u(j) + \sum_{j=0}^{n} w(j)u^p(j), \qquad n = 0,1,\ldots$$

implies that

$$e(n)u(n+1) \leq \left[u_0^{1-p} + (1-p) \sum_{k=0}^{n} w(k)e^{1-p}(k)\right]^{\frac{1}{1-p}}, \qquad n = 0,1,\ldots,$$

where

$$e(n) = \prod_{j=0}^{n} (1 + v(j))^{-1}, \quad n = 0,1,\ldots .$$

Lemma 5. (Hardy, Littlewood and Polya [11, p. 26]). *Let* $a_i \geq 0$ *for* $i = 1,\ldots,m$. *Then*

$$\sum_{i=1}^{m} a_i^p \leq m^{1-p} \left(\sum_{i=1}^{m} a_i\right)^p \quad \textit{for } 0 \leq p \leq 1,$$

$$m^{p-1} \sum_{i=1}^{m} a_i^p \geq \left(\sum_{i=1}^{m} a_i\right)^p \quad \textit{for} \quad p \geq 1.$$

3. RESULTS

We are now able to give our main result.

Theorem 1. *Suppose that exists* $M > 0$ *such that*

$$\prod_{m=0}^{s} |p_0(m)| \geq M$$

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and that all the solutions of (3) are in ℓ^p. *If* $q_i \in \ell^k$ *where*

$$1 \le k \le \infty \text{ for } 1 \le p \le 2$$

and

$$1 \le k \le \frac{p}{p-2} \text{ for } 2 < p \le \infty,$$

then all the solutions of (4) are in ℓ^p, *too.*

Proof. Starting with the representation given by Lemma 1 and using the fact that $G(s,m) = 0$ for $s = m-n+1,\ldots,m-1$, we get

$$\Delta^j y(m) = \sum_{i=1}^{n} c_i \Delta^j y_i(m) + \sum_{i=1}^{n} \Delta^j y_i(m) \sum_{s=0}^{m-1} (-1)^{n+i} W_i(s) \frac{f(s)}{D(0) \prod_{k=0}^{s} p_0(k)}, \tag{5}$$

for $m = 0,1,\ldots$ and $j = 0,\ldots,m-1$.

Obviously,

$$|W_i(s)| \le C \sum_{\substack{j=1 \\ j\neq i}}^{n} |y_j(s+1)| \le C \sum_{j=1}^{n} |y_j(s+1)|,$$

for every $i = 1,\ldots,n$ and some $C > 0$. So, by (5) we get

$$|\Delta^j y(m)| \le C_1 \phi_j(m)\Big[1 + \sum_{s=0}^{m-1} \phi_0(s+1)|f(s)|\Big], \tag{6}$$

$$j = 0,\ldots,n-1; \quad m = 0,1,\ldots,$$

where we denote $\phi_j(m) = \max\{|\Delta^j y_i(m)| : i = 1,\ldots,n\}$ for $j = 0,\ldots,n-1$.

As

$$f(s) = -\sum_{i=0}^{n-1} q_i(s) y(i+s) = \sum_{i=0}^{n-1} \bar{q}_i(s) \Delta^i y(s)$$

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where $\bar{q}_i(s)$ are linear combinations of $q_i(a)$, relation (6) leads to

$$(7) \qquad |\Delta^j y(m)| \le C_1 \phi_j(m)\Big[1 + \sum_{s=0}^{m-1} \phi_0(s+1)q(s) \sum_{i=0}^{n-1} |\Delta^i y(s)|\Big],$$

$$j = 0,\dots,n-1; \quad m = 1,2,\dots,$$

where $q(s) = max\{|\bar{q}_i(s)|, i = 0,\dots,n-1\}$.

We shall put

$$\frac{1}{p} + \frac{1}{p'} = \frac{1}{k} + \frac{1}{k'} = 1$$

and differ the next cases:

a) $1 \le k \le p'$.

Relation (7) and the boundedness of the solutions of equation (3) imply

$$\sum_{j=0}^{n-1} |\Delta^j y(m)| \le C_2\Big[1 + \sum_{s=0}^{m-1} \phi_0(s+1)q(s) \sum_{i=0}^{n-1} |\Delta^i y(s)|\Big],$$

$$m = 1,2,\dots .$$

$\phi_0 \in \ell^p$ implies $\phi_0 \in \ell^{k'}$ and because of $q \in \ell^k$, $\phi\, q \in \ell$ such that

$$\sum_{j=0}^{n-1} |\Delta^j y(m)|$$

is bounded according to Lemma 4.

Using again relation (7), we obtain

$$|\Delta^j y(m)| \le C_1 \phi_j(m)\Big[1 + C_3 \sum_{s=0}^{n-1} \phi_0(s+1)q(s)\Big] \le$$

$$\le C_4 \phi_j(m), \quad j = 0,\dots,n-1; \quad m = 1,2,\dots$$

which implies that $\Delta^j y \in \ell^p$, $j = 0,\dots,n-1$.

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b) $1 \le p \le 2 \le p' < k$

Using Hölder's inequality and Lemma 5, we get

$$\sum_{s=0}^{m-1} \phi_0(s+1)q(s) \sum_{i=0}^{n-1} |\Delta^j y(s)| \le$$

$$\le \Big(\sum_{s=0}^{m-1} \phi_0(s+1)q(s))^{p'} \Big)^{1/p'} \Big(\sum_{s=0}^{m-1} \Big(\sum_{i=0}^{n-1} |\Delta^i y(s)|)^p \Big)^{\frac{1}{p}} \le$$

$$\le C_5 \Big(\sum_{s=0}^{m-1} \phi_0^{p'}(s+1)) \Big)^{1/p'} \Big(\sum_{s=0}^{m-1} \sum_{i=0}^{n-1} |\Delta^i y(s)|^p \Big)^{1/p} \le$$

$$\le C_6 \Big(\sum_{s=0}^{m-1} \phi_0^p(s+1)) \Big)^{1/p'} \Big(\sum_{s=0}^{m-1} \sum_{i=0}^{n-1} |\Delta^i y(s)|^p \Big)^{1/p} \le$$

$$\le C_7 \Big(\sum_{s=0}^{m-1} \sum_{i=0}^{n-1} |\Delta^i y(s)|^p \Big)^{1/p}.$$

Putting the last inequality in relation (7), we obtain

$$|\Delta^j y(m)| \le C_1 \phi_j(m) \Big[1 + C_7 \Big(\sum_{s=0}^{m-1} \sum_{i=0}^{n-1} |\Delta^i y(s)|^p \Big)^{1/p} \Big],$$

$$j = 0,\dots,n-1;\ m = 1,2,\dots ,$$

which, according to Lemma 5, implies

$$\sum_{j=0}^{n-1} |\Delta^j y(m)|^p \le C_8 \sum_{j=0}^{n-1} \phi_j^p(m) \Big[1 + \sum_{s=0}^{m-1} \sum_{i=0}^{n-1} |\Delta^i y(s)|^p \Big], \tag{8}$$

$$m = 1,2,\dots .$$

Summing (8) from 0 to r-1 and using the fact that $\phi_j^p \in \ell$, Lemma 4 implies that all the solutions of (4) are in ℓ^p.

c) $1 < p' < 2 < p, \quad p' < k \le \frac{p}{p-2}$.

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Now,

$$\sum_{s=0}^{m-1} (\phi_0(s+1)q(s))^{p'} \le$$

$$\le \Big(\sum_{s=0}^{m-1} \phi_0^p(s+1) \Big)^{1/p} \Big(\sum_{s=0}^{m-1} (q(s))^{\frac{pp'}{p-p'}} \Big)^{\frac{p-p'}{pp'}} \le$$

$$\le C_9 \Big(\sum_{s=0}^{m-1} \phi_0^p(s+1) \Big)^{1/p} \Big(\sum_{s=0}^{m-1} q^k(s) \Big)^{\frac{p-p'}{pp'}} \le C_{10},$$

and the proof follows the same line as in case b).

If p,k satisfy the same conditions as in Theorem 1, we are able to give a converse result as follows.

Theorem 2. *Suppose that*

$$\prod_{m=0}^{s} |p_0(m)| \ge M > 0$$

and $q_i \in \ell^k$ *for* $i = 0,\dots,n-1$. *If some solution of (3) is not in* ℓ^p *then the same in true for some solution of (4).*

Proof. Suppose that all the solutions of (4) are in ℓ^p, which by Theorem 1 implies that all the solutions of (3) are in ℓ^p, since equation (3) can be considered as a linear perturbation of (4) i.e.

$$Dy = D_L y - \sum_{i=0}^{n-1} q_i(m)y(m+i).$$

The linear differential equation of the form (3) is of special interest when $p_{n-1}(t) \equiv 0$. In the theory of the difference equation, the adequate case is when $p_0(m) \equiv 1$. For that case, when operators D and D_L take, respectively, the form

$$(9) \qquad My \equiv y(n+m) + p_{n-1}(m)y(n+m-1) + \dots + p_1(m)y(m+1) + y(m) = 0, \quad m = 0,1,\dots$$

and

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$$M_L y \equiv My + q_{n-1}(m)y(n+m-1) + \ldots + q_1(m)y(m+1) = 0, \quad m = 0,1,\ldots, \tag{10}$$

we obtain from Theorem 1 and 2 the following

Corollary. *Suppose that* p,k *satisfy the same conditions as in Theorem 1. Then, all the solutions of equation (9) are in* ℓ^p *iff the same is true for all the solutions of (10).*

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REZIME

PERTURBACIJE LINEARNE DIFERENCNE JEDNAČINE

U radu se posmatra linearna diferencna jednačina oblika

$$(1) \qquad Dy = y(n+m) + p_{n-1}(m)y(n+m-1) + \ldots + p_0(m)y(m) = 0, \quad m = 0,1,\ldots$$

i njoj pridružena linearna perturbacija oblika

$$(2) \qquad D_L y = Dy + q_{n-1}(m)y(n+m-1) + \ldots + q_0(m)y(m) = 0, \quad m = 0,1,\ldots .$$

Ukoliko nizovi q_i pripadaju prostoru ℓ^k a sva rešenja jednačine (1) prostoru ℓ^p, uz odredjenu zavisnost parametara k i p, sva rešenja jednačine (2) takodje pripadaju ℓ^p.

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A COMMON FIXED POINT THEOREM IN UNIFORMIZABLE SPACES

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ABSTRACT

In this paper a generalization of Theorem 1 from [1] in uniformizable spaces is proved.

Let us recall Theorem 1 from [1].

THEOREM A. *Let* (X,d) *be a complete metric space,* S *and* T *one to one continuous mappings from* X *into* X, A *a continuous mapping from* X *into* $SX \cap TX$ *and* A *commute with* S *and* T.

Suppose that the following conditions are satisfied:

1) For every $x \in X$ *there exists* $n(x) \in \mathbb{N}$ *so that for every* $y \in X$:

$$d(A^{n(x)}x, A^{n(x)}y) \leq \min\{q[d(Sx,Ty)]d(Sx,Ty), d(Tx,Sy)\}$$

where $q: [0,\infty) \to [0,1)$ *is a nondescreasing function such that*

$$\lim_{t\to\infty} t(1 - q(t)) = +\infty .$$

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2) For some $x_0 \in X$ *one of the sets:*

$$\{A^m T^p(x_0): p \in \mathbb{N}, m \in \{0,1,\ldots,n(x_0)-1\}$$

and

$$\{A^m S^p(x_0): p \in \mathbb{N}, m \in \{0,1,\ldots,n(x_0)-1\}$$

is bounded.

Then there exists one and only one element $y \in X$ *such that:*

$$y = Sy = Ty = Ay .$$

Further, we shall recall the definition of a uniformizable space.

Let X be an arbitrary set. A mapping $d: X \times X \to \mathbb{R}^+ = [0,\infty)$ is called a pseudometric iff for every $x,y,z \in X$

1. $d(x,y) \geq 0$; $d(x,x) = 0$;
2. $d(x,y) = d(y,x)$;
3. $d(x,y) \leq d(x,z) + d(z,y)$.

A pair $(X,\{d_i\}_{i\in I})$, where d_i is a pseudometric for every $i \in I$, is called a uniformizable space.

The convergence of the sequences, the notions of a Cauchy sequence and completeness $(X,\{d_i\})_{i\in I}$ are introduced in the usual way [3].

Now, we shall prove a generalization of Theorem A in uniformizable spaces.

THEOREM. *Let* $(X,\{d_i\}_{i\in I})$ *be a complete Hausdorff uniformizable space,* $f: I \to I$, S *and* T *one to one continuous mappings from* X *into* X, $A: X \to SX \cap TX$ *be continuous so that* A *commutes with* S *and* T *and the following conditions are satisfied:*

1) For every $x \in X$ *there exists* $n(x) \in \mathbb{N}$ *so that for every* $y \in X$ *and every* $i \in I$:

$$d_i(A^{n(x)}x, A^{n(x)}y) \leq$$

$$\leq \min q_i\{d_{f(i)}(Sx,Ty)[d_{f(i)}(Sx,Ty)]d_{f(i)}(Tx,Sy)\}$$

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where $q_i : [0,\infty) \to [0,1)$ *is a nondecreasing function for every* $i \in I$ *so that for every* $t \in [0,\infty)$:

$$\sup_{k\in N_0} q_{f^k(i)}(t) < 1 \qquad (N_0 = N \cup \{0\})$$

and

$$\lim_{t\to+\infty} [t - t \sup_{k\in N_0} q_{f^k(i)}(t)] = +\infty.$$

2) There exists $x_0 \in X$ *so that for every* $i \in I$ *there is* $L^{(i)}$ *or* $\tilde{L}^{(i)}$ *so that:*

a) $$L^{(i)} \geq \max\left\{\sup_{k\in N_0} d_{f^k(i)}\left(A^{n(x_0)}x_0, Sx_0\right), \sup_{\substack{\ell,k\in N_0 \\ 0\leq r<n(x_0)}} d_{f^k(i)}\left(A^r T^\ell x_0, Sx_0\right)\right\}$$

or

b) $$\tilde{L}^{(i)} \geq \max\left\{\sup_{k\in N_0} d_{f^k(i)}\left(A^{n(x_0)}x_0, Tx_0\right), \sup_{\substack{\ell,k\in N_0 \\ 0\leq r<n(x_0)}} d_{f^k(i)}\left(A^r S^\ell x_0, Tx_0\right)\right\}.$$

Then there exists $y \in X$ *so that* $y = Ay = Sy = Ty$.

This common fixed point is unique with the property that the sequence

$$\{d_{f^n(i)}(y, Sx_0)\}_{n\in N}$$

is bounded for every $i \in I$.

PROOF. Since $AX \subseteq SX \cap TX$ (as in [1]), we can define the sequence $\{x_n\}_{n\in N_0}$, where x_0 is from the condition 2)a. of the Theorem, such that:

$$Sx_{2k} = A^{n(x_{2k-1})}x_{2k-1}, \quad Tx_{2k-1} = A^{n(x_{2k-2})}x_{2k-2},$$

$k \in N$. Let

$$y_m = \begin{cases} Tx_{2k-1} & m = 2k-1 \\ Sx_{2k} & m = 2k. \end{cases}$$

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Further, for every $m \in \mathbb{N}_0$ and every $\ell \in \mathbb{N}$ we have:

$$d_{f^m(i)}\left(A^{n(x_{2k-1})}x_{2k-1}, A^{\ell}x_{2k}\right) =$$

$$= d_{f^m(i)}\left(A^{n(x_{2k-1})}x_{2k-1}, A^{\ell}S^{-1}Sx_{2k}\right) =$$

$$= d_{f^m(i)}\left(A^{n(x_{2k-1})}x_{2k-1}, A^{n(x_{2k-1})}S^{-1}A^{\ell}x_{2k-1}\right) \le$$

$$\le d_{f^{m+1}(i)}\left(Tx_{2k-1}, A^{\ell}x_{2k-1}\right) =$$

$$= d_{f^{m+1}(i)}\left(A^{n(x_{2k-2})}x_{2k-2}, A^{\ell}x_{2k-1}\right) =$$

$$= d_{f^{m+1}(i)}\left(A^{n(x_{2k-2})}x_{2k-2}, A^{\ell}T^{-1}Tx_{2k-1}\right) =$$

$$= d_{f^{m+1}(i)}\left(A^{n(x_{2k-2})}x_{2k-2}, A^{n(x_{2k-2})}T^{-1}A^{\ell}x_{2k-2}\right) \le$$

$$\le q_{f^{m+1}(i)}\left[d_{f^{m+2}(i)}\left(Sx_{2k-2}, A^{\ell}x_{2k-2}\right)\right] \cdot$$

$$\cdot d_{f^{m+2}(i)}\left(Sx_{2k-2}, A^{\ell}x_{2k-2}\right) =$$

$$= q_{f^{m+1}(i)}\left[d_{f^{m+2}(i)}\left(A^{n(x_{2k-3})}x_{2k-3}, A^{\ell}x_{2k-2}\right)\right] \cdot$$

$$\cdot d_{f^{m+2}(i)}\left(A^{n(x_{2k-3})}x_{2k-3}, A^{\ell}x_{2k-2}\right) \le \ldots \le$$

$$(1) \qquad \le \prod_{s=1}^{k} q_{f^{m+2s-1}(i)}\left[d_{f^{m+2k}(i)}\left(Sx_0, A^{\ell}x_0\right)\right] \cdot$$

$$\cdot d_{f^{m+2k}(i)}\left(Sx_0, A^{\ell}x_0\right)$$

Similarly:

$$d_{f^m(i)}\left(A^{n(x_{2k})}x_{2k}, A^{\ell}x_{2k+1}\right) \le$$

$$(1') \qquad \le \prod_{s=0}^{k} q_{f^{m+2s}(i)}\left[d_{f^{m+2k+1}(i)}\left(Sx_0, A^{\ell}x_0\right)\right] \cdot$$

$$\cdot d_{f^{m+2k+1}(i)}\left(Sx_0, A^{\ell}x_0\right)$$

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so,

$$d_i(y_{2k}, y_{2k+1}) \le \prod_{s=1}^{k} q_{f^{2s-1}(i)} \left[d_{f^{2k}(i)}(Sx_o, A^{n(x_{2k})} x_o) \right] \cdot \tag{2}$$

$$\cdot\, d_{f^{2k}(i)}(Sx_o, A^{n(x_{2k})} x_o)$$

and

$$d_i(y_{2k-1}, y_{2k}) \le \prod_{s=0}^{k} q_{f^{2s}(i)} \left[d_{f^{2k-1}(i)}(Sx_o, A^{n(x_{2k-1})} x_o) \right] \cdot \tag{2'}$$

$$\cdot\, d_{f^{2k-1}(i)}(Sx_o, A^{n(x_{2k-1})} x_o) .$$

Let us prove that the sequence

$$\{ d_{f^k(i)}(Sx_o, A^m T^\ell x_o) \} \qquad m,k,\ell \in N_o$$

is bounded.

Since x_o is from condition 2.a, for every $i \in I$ there exists:

$$b_o^{(i)} = \max \Big\{ \sup_{k \in N_o} d_{f^k(i)}(A^{n(x_o)} x_o, Sx_o), \sup_{\substack{\ell,k \in N_o \\ 0 \le r < n(x_o)}} d_{f^k(i)}(A^r T^\ell x_o, Sx_o) \Big\} .$$

Let $m = p \cdot n(x_o) + r$, where $0 \le r < n(x_o)$. We shall prove that for every $i \in I$ there exists a bounded sequence $\{b_p^{(i)}\}_{p \in N_o}$ such that:

$$d_{f^k(i)}(A^{p \cdot n(x_o)+r} T^\ell x_o, Sx_o) \le b_p^{(i)} \tag{3}$$

for every p,ℓ,k in N_o where:

$$b_p^{(i)} = \sup_{k \in N_o} q_{f^k(i)} \left[b_{p-1}^{(i)} \right] \cdot b_{p-1}^{(i)} + b_o^{(i)}, \quad p \in N.$$

The proof of (3) will be given by induction in respect to $p \in N$.

For $p = 1$ we have that:

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$$d_{f^{k}(i)}(A^{n(x_0)+r}\,T^{\ell}x_0,\ Sx_0) \le$$

$$\le\ q_{f^{k}(i)}[(d_{f^{k+1}(i)}(A^{r}\,T^{\ell+1}x_0,\ Sx_0)]\cdot$$

$$\cdot\ d_{f^{k+1}(i)}(A^{r}\,T^{\ell+1}x_0,\ Sx_0) + d_{f^{k}(i)}(A^{n(x_0)}x_0, Sx_0)$$

so

$$\sup_{k\in N_0} d_{f^{k}(i)}(A^{n(x_0)+r}T^{\ell}x_0,\ Sx_0) \le$$

$$\le \sup_{k\in N_0} q_{f^{k}(i)}[b_0^{(i)}]\ b_0^{(i)} + b_0^{(i)} = b_1^{(i)}\ .$$

Suppose that (3) is satisfied for some $p \in N$ and for every $\ell \in N_0, k \in N_0$ and prove that:

$$d_{f^{k}(i)}(A^{(p+1)n(x_0)+r}\,T^{\ell}x_0,\ Sx_0) \le b_{p+1}^{(i)}$$

for every $\ell,\ k \in N_0$.

$$d_{f^{k}(i)}(A^{(p+1)n(x_0)+r}\,T^{\ell}x_0,\ Sx_0) \le$$

$$\le\ q_{f^{k}(i)}[d_{f^{k}(i)}(A^{p\ n(x_0)+r}\,T^{\ell+1}x_0,\ Sx_0)]\cdot$$

$$\cdot\ d_{f^{k+1}(i)}(A^{p\cdot n(x_0)+r}\,T^{\ell+1}x_0,\ Sx_0) +$$

$$+\ d_{f^{k}(i)}(A^{n(x_0)}x_0,\ Sx_0) \le$$

$$\le\ q_{f^{k}(i)}\left[b_p^{(i)}\right]\cdot b_p^{(i)} + b_0^{(i)} \le$$

$$\le\ \sup_{k\in N_0} q_{f^{k}(i)}\left[b_p^{(i)}\right]\cdot b_p^{(i)} + b_0^{(i)} = b_{p+1}^{(i)}\ .$$

Since all the functions q_i are nondecreasing, $\sup_{k\in N_0} q_{f^{k}(i)}$ is nondecreasing too, so similarly as in [1] (using Lemma 2.2 [5])

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we have that $\left\{b_p^{(i)}\right\}_{p\in N}$ is bounded for every $i \in I$ and let

$$D_i = \sup_{\substack{k,\ell\in N_o \\ m\in N}} d_{f^k(i)}(A^m T^\ell x_o, Sx_o).$$

Now we have that:

$$d_i(y_{2k}, y_{2k+1}) \le \prod_{s=1}^{k} q_{f^{2s-1}(i)}^{[D_i]} D_i \le$$

$$\le \left(\sup_{j\in N_o} q_{f^j(i)}^{[D_i]}\right)^k D_i$$

and

$$d_i(y_{2k-1}, y_{2k}) \le \prod_{s=0}^{k-1} q_{f^{2s}(i)}^{[D_i]} \cdot D_i \le$$

$$\le \left(\sup_{j\in N_o} q_{f^j(i)}^{[D_i]}\right)^k D_i$$

which implies for $m = 2k$ (and for the other case as well) that:

$$d_i(y_m, y_{m+k}) \le \sum_{j=1}^{k} d_i(y_{m+j-1}, y_{m+j}) \le$$

$$\le \left(\sup_{j\in N_o} q_{f^j(i)}^{[D_i]}\right)^k D_i + \left(\sup_{j\in N_o} q_{f^j(i)}^{[D_i]}\right)^{k+1} D_i + \ldots.$$

Since

$$\sup_{j\in N_o} q_{f^j(i)}^{[D_i]} \le q(i) < 1$$

$\{y_n\}$ is a Cauchy sequence and let $y = \lim_{n\to\infty} y_n$.

Further, $\{Sx_{2k}\}_{k\in N}$ and $\{Tx_{2k-1}\}_{k\in N}$ are subsequences of sequence $\{y_n\}_{n\in N}$ so it follows that:

$$\lim_{k\to\infty} Sx_{2k} = \lim_{k\to\infty} Tx_{2k-1} = y .$$

Also, using the inequality (1) for $\ell = 1,2$ we obtain that:

$$d_i(Sx_{2k}, Ax_{2k}) = d_i(A^{n(x_{2k-1})} x_{2k-1}, Ax_{2k}) \le$$

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$$\leq \prod_{s=1}^{k} q_{f^{2s-1}(i)} [d_{f^{2k}(i)}(Sx_o, Ax_o)] \cdot d_{f^{2k}(i)}(Sx_o, Ax_o) \leq$$

$$\leq \left(\sup_{j \in N_o} q_{f^j(i)} [D_i] \right)^k \cdot D_i$$

and

$$d_i(Sx_{2k}, A^2x_{2k}) = d_i(A^{n(x_{2k-1})} x_{2k-1}, A^2x_{2k}) \leq \ldots \leq$$

$$\leq \left(\sup_{j \in N_o} q_{f^j(i)} [D_i] \right)^k D_i$$

which implies that:

$$\lim_{k \to \infty} Ax_{2k} = \lim_{k \to \infty} A^2x_{2k} = \lim_{k \to \infty} Sx_{2k} = y .$$

From the continuity of A and S and the commutativity we obtain that:

$$Ay = A(\lim_{k \to \infty} Sx_{2k}) = S(\lim_{k \to \infty} Ax_{2k}) = Sy$$

and

$$y = \lim_{k \to \infty} A^2x_{2k} = A(\lim_{k \to \infty} Ax_{2k}) = Ay.$$

Similarly, it follows that Ay = Ty and so we have that y is a common fixed point for mappings A, S and T.

Since:

$$d_{f^n(i)}(y_m, Sx_o) \leq \sum_{j=1}^{m-1} d_{f^n(i)}(y_j, y_{j+1}) + d_{f^n(i)}(y_1, Sx_o)$$

using inequalities (2) and (2′) we have that:

$$d_{f^n(i)}(y, Sx_o) \leq M_i < +\infty$$

for every $n \in N_o$ so:

$$\sup_{n \in N_o} d_{f^n(i)}(y, Sx_o) \leq M_i \quad \text{for every } i \in I.$$

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Now, suppose that there exists an another common fixed point z for A, S and T such that for any $i \in I$ and every $k \in N_o$:

$$d_{f^k(i)}(z, Sx_o) \leq Q_i \cdot$$

It follows that for every $k \in N_o$:

$$d_{f^k(i)}(y,z) \leq T_i + Q_i$$

which implies that:

$$d_i(y,z) \leq q_i[d_{f(i)}(y,z)] \cdot d_{f(i)}(y,z) \leq$$

$$\leq q_i[T_i + Q_i] \cdot d_{f(i)}(y,z) \leq \ldots \leq$$

$$\leq \left(\sup_{n \in N_o} q_{f^n(i)}\right)^n [T_i + Q_i] (T_i + Q_i)$$

for every $i \in I$. Since $\sup_{n \in N_o} q_{f^n(i)} (T_i + Q_i) \leq q < 1$, we obtain that:

$$d_i(y,z) = 0 ,$$

for every $i \in I$ and so:

$$y = z.$$

Corollary. *Let* $(X, \{d_i\}_{i \in I})$ *be a complete Hausdorff uniformizable space,* $f : I \to I$, $A : X \to X$ *satisfying the following conditions:*

1. *For every* $i \in I$ *there exists* $q_i \geq 0$ *such that:*

$$d_i(Ax, Ay) \leq q_i d_{f(i)}(x,y) \text{ for all } x,y \in X,$$

and for every $i \in I$:

$$\sup_{k \in N_o} q_{f^k(i)} \leq q^{(i)} < 1.$$

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2. *There exists* $x_0 \in X$ *such that for every* $i \in I$ *sequence*

$$\left\{d_{f^{n(i)}}(x_0, Ax_0)\right\}_{n \in N}$$

is bounded.

Then there exists one and only one solution of the equation $y = Ay$ *which also satisfies the condition:*

$$d_{f^{n(i)}}(y, x_0) \leq p(i) < +\infty \quad n \in N.$$

Proof. For $S = T = Id$, $n(x) = 1$ for every $x \in X$ and $q_i(t) = q_i$ for every $t \in [0,\infty)$ all the conditions are satisfied so Theorem conclude our statement. If d_i, $i \in I$, are seminorms this corollary is, in fact, theorem 1 from [2].

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REZIME

TEOREMA O ZAJEDNIČKOJ NEPOKRETNOJ TAČKI U UNIFORMIZABILNIM PROSTORIMA

U ovom radu dokazano je uopštenje teoreme 1 iz [1] u uniformizabilnim prostorima. Slični rezultati su dobijeni u [2] i [3].

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A NOTE ON MAPPINGS AND PARACOMPACTNESS

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ABSTRACT

In the note some properties of paracompactness, α-regular and α-Hausdorff subsets, closed, almost closed and continuous mappings are observed.

1. PRELIMINARIES

Throughout the present paper, spaces will always mean topological spaces on which no separation axioms are assumed, unless explicitly stated.

A space X is *paracompact* iff every open covering of X has an open locally finite refinement, [3].

A space X is *almost paracompact* iff for every open covering $\mathcal{U}$ of X there exists an open locally finite family $\mathcal{V}$ which refines $\mathcal{U}$ such that $X = \cup\{\overline{V} : V \in \mathcal{V}\}$, [10].

A subset A of a space X is *regularly open (regularly closed)* iff it is the interior (closure) of some closed (open) set, or equivalently iff it is the interior (closure) of its own closure (interior), [2].

A subset A of a space X is *α-paracompact (α-nearly*

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paracompact) iff for every X-open (X-regularly open) cover $\mathcal{U}$ of A there exists an X-open X-locally finite family $\mathcal{V}$ which refines $\mathcal{U}$ and covers A, [13], ([8]).

A subset A of a space X is *α-almost paracompact* iff for every X-open cover of A there exists an X-locally finite family of X-open sets which refines it and the X-closures of whose members cover the set A, [5]. A subset A of a space X is *α-nearly compact* iff every X-regularly open cover of A has a finite subcovering, [11].

A subset A of a space X is *α-Hausdorff* iff any two points a,b of X, where $a \in A$ and $b \in X \setminus A$, can be strongly separated, [9]. A subset A of a space X is *α-regular* iff for any point $a \in A$ and any open set U containing a there exists an open set V such that $a \in V \subset \overline{V} \subset U$ or equivalently, for any closed set F of a space X and any point $x \in A$ such that $x \in X \setminus F$, there exist disjoint neighbourhoods of x and F, respectively, [9].

The multifunction $F : X \to Y$ is *closed (almost closed)* iff for any closed (regularly closed) set $A \subset X$, $F(A) = \cup \{F(x) : x \in A\}$ is closed in Y, [1], ([6]).

Theorem 1.1. ([7]) *Let* X *be a space such that there exists a dense α-regular subset* D. *If every* X*-open cover of* D *has an* X*-locally finite refinement which covers* D, *then* D *is α-paracompact, i.e.* X *is paracompact.*

Theorem 1.2. ([9]) *Let* A *be any α-regular α-paracompact subset of a space* X. *For each open neighbourhood* U *of* A, *there exists an open neighbourhood* V *of* A *such that* $A \subset V \subset \overline{V} \subset U$.

Theorem 1.3. ([5]) *A space* X *is almost paracompact iff for every regularly open covering* $\mathcal{U}$ *of* X *there exists an open locally finite family* $\mathcal{V}$ *which refines* $\mathcal{U}$ *such that* $X = \cup \{\overline{V} : V \in \mathcal{V}\}$.

Theorem 1.4. ([6]) *A surjective multifunction*

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$F : X \to Y$ is almost closed iff for any subset $S \subset Y$ and any regularly open set U of a space X containing $F^-(S)$ $(F^-(S) = \{x \in X : F(x) \cap S \neq \emptyset\})$, there exists an open set V in Y such that $S \subset V$ and $F^-(V) \subset U$.

Theorem 1.5. ([6]) *Let a multifunction $F : X \to Y$ be an almost closed surjection such that $F^-(y)$ is α-nearly compact for each point $y \in Y$. If $\mathcal{U} = \{U_i : i \in I\}$ is an open locally finite family then $F(\mathcal{U}) = \{F(U_i) : i \in I\}$ is a locally finite family.*

2. SUBSETS AND PARACOMPACTNESS

Lemma 2.1. *Let A be any dense subset of a space X such that every open covering of A is an open covering of X. If X is paracompact then A is α-paracompact.*

Proof. Obvious.

In this lemma we supposed that every open covering of A is an open covering of X. If every open covering of a dense subset A is not an open covering of the space X, then A is not always α-paracompact, as can be seen from the following example.

Example 2.1. Let

$$X = \{a_i, b_i, a : i = 1,2,\ldots\} .$$

Let each point a_i be isolated. Let each point b_i be isolated. Let the fundamental system of neighbourhoods of a be the set

$$\{V^n(a) : n = 1,2,\ldots\}$$

where

$$V^n(a) = \{a, a_i : i \geq n\} .$$

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Let

$$A = \{a_i, b_i : i = 1,2,\ldots\}.$$

The space X is a Hausdorff regular paracompact. A is not α-paracompact.

Lemma 2.2. *Let* A *be any dense* α*-regular subset of a space* X *such that every open covering of* A *is an open covering of* X. *If* X *is almost paracompact, then* X *is paracompact.*

Proof. Let

$$U = \{U_i : i \in I\}$$

be any open covering of X. For each point $x \in A$, there exists an open set V_x such that

$$x \in V_x \subset \overline{V}_x \subset U_i$$

for some $i \in I$.

Now,

$$V = \{V_x : x \in A\}$$

is an open covering of A, i.e. of X. Since X is almost paracompact, there exists an open locally finite family

$$W = \{W_j : j \in J\}$$

which refines V such that

$$X = \overline{\cup\{W_j : j \in J\}} = \cup\{\overline{W}_j : j \in J\}.$$

Now, for each $j \in J$, there exists an $x(j) \in A$ such that

$$W_j \subset V_{x(j)} \subset \overline{V}_{x(j)} \subset U_{i(x)}$$

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for some $i(x) \in I$.
Hence, we have

$$\overline{W}_j \subset \overline{V}_{x(j)} \subset U_{i(x)} .$$

Now, the collection

$$\{\overline{W}_j : j \in J\}$$

is a locally finite family which refines U and covers A, hence A is α-paracompact, i.e. X is paracompact.

Theorem 2.1. *Let* A *be a dense* α*-regular subset of a space* X *such that every open covering of* A *is an open covering of* X. *If* X *is almost paracompact, then* A *is* α*-paracompact.*

Proof. It follows easily from Lemma 2.1 and Lemma 2.2.

Corollary 2.1. *Every regular almost paracompact is paracompact.*

There exists a space with the properties as in Lemma 2.2 which is not regular (Example 2.3 in [7]).

In Lemma 2.2 we supposed that every open covering of A is an open covering of X. If there is an open covering of a dense subset A which is not an open covering of X, then X need not be paracompact, as can be seen from the following example.

Example 2.2. Let

$$X = \{a_{ij}, a_i, a : i,j = 1,2,\ldots\}.$$

Let each point $a_{i,j}$ be isolated.
Let the fundamental system of neighbourhoods of a_i be the set

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$$\{U^n(a_i) : n = 1,2,...\}$$

where

$$U^n(a_i) = \{a_i, a_{ij} : j \geq n\} .$$

Let the fundamental system of neighbourhoods of a be the set

$$\{V^n(a) : n = 1,2,...\}$$

where

$$V^n(a) = \{a, a_{ij} : i \geq n, j \geq n\} .$$

Then X is a Hausdorff space which is not regular at a and hence X is not paracompact (every Hausdorff paracompact space is regular). But X is almost paracompact. The subset

$$A = \{a_{ij}, a_i : i,j = 1,2,...\}$$

is a dense α-regular subset of the space X. Not every open covering of A is an open covering of X (the set A is open).

Theorem 2.2. *Every α-Hausdorff α-almost paracompact subset of a space X is closed.*

Proof. Let A be any α-Hausdorff α-almost paracompact subset and a be any point of $X \setminus A$. For each $x \in A$, there exist open sets U_x and V_x such that

$$x \in U_x, \ a \in V_x \text{ and } U_x \cap V_x \neq \emptyset.$$

Now,

$$\mathcal{U} = \{U_x : x \in A\}$$

is an open covering of A. Since A is α-almost paracompact,

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then there exists an X-open X-locally finite family

$$W = \{W_j : j \in J\}$$

which refines U such that

$$A \subset \overline{\cup\{W_j : j \in J\}} = \cup\{\overline{W}_j : j \in J\}.$$

Since W is X-locally finite, there exists an open neighbourhood U of a and a finite subset J_o of J such that

$$U \cap W_j \neq \emptyset \text{ for } j \in J_o \text{ and } U \cap W_j = \emptyset \text{ for } j \in J \setminus J_o.$$

For each $j \in J_o$ there is an $x(j) \subset A$ such that $W_j \subset U_{x(j)}$.

Let

$$U_1 = U \cap (\cap\{V_{x(j)} : j \in J_o\}).$$

U_1 is an open neighbourhood of a such that

$$a \in U_1 \subset X \setminus A$$

hence $X \setminus A$ is open, i.e. A is closed.

Corollary 2.2. *Let* A *be any dense* α*-almost paracompact* α*-Hausdorff subset of a space* X. *Then* A = X, *that is in the space* X, *there is no proper* α*-Hausdorff* α*-almost paracompact dense subset.*

3. MAPPINGS AND SUBSETS

Theorem 3.1. *Let* f *be a closed and continuous mapping of a space* X *onto a space* Y *and let* A *be a non-empty subset of* X. *If for each* $x \in A \subset X$ *the set* $f^{-1}(f(x))$ *is* α*-regular and* α*-paracompact, then* f(A) *is* α*-regular.*

Proof. Let $y \in f(A)$ and V be an open set containing

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y. By Theorem 1.2 there exists an open set U in X such that

$$f^{-1}(y) \subset U \subset \overline{U} \subset f^{-1}(V).$$

Since f is closed, there exists an open set W in Y such that $y \in W$ and $f^{-1}(W) \subset U$. Now, we have

$$y \in W \subset f(U) \subset f(\overline{U}) \subset V.$$

Hence,

$$y \in W \subset \overline{W} \subset f(\overline{U}) \subset V$$

i.e. f(A) is α-regular.

Corollary 3.1. *If f is a closed and continuous mapping of a regular space X onto a space Y such that* $f^{-1}(y)$ *is α-paracompact for each point* $y \in Y$, *then Y is regular.*

Problem 3.1. Let f be a closed and continuous mapping of a space X onto a space Y such that $f^{-1}(f(x))$ is α-paracompact for each $x \in A$. If A is α-regular, is f(A) α-regular?

Theorem 3.2. *Let f be any almost closed mapping of a space X onto a space Y. Let B be a closed subset of X such that for each* $x \in X \setminus B$ *the set* $f^{-1}(f(x))$ *is α-regular and α-paracompact. Then f(B) is closed.*

Proof. Let

$$y \in Y \setminus f(B), \text{ i.e. } f^{-1}(y) \subset X \setminus B.$$

By Theorem 1.2 there exists an open neighbourhood V of $f^{-1}(y)$ such that

$$f^{-1}(y) \subset V \subset \overline{V} \subset X \setminus B.$$

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Since f is almost closed, there exists an open set W of Y such that

$$y \in W$$

and

$$f^{-1}(y) \subset f^{-1}(W) \subset \alpha(V) = \overline{V}^{\circ} \subset X \setminus B.$$

Hence, we have

$$y \in W \subset Y \setminus f(B)$$

and thus $Y \setminus f(B)$ is open, i.e. $f(B)$ is closed.

4. MAPPINGS AND PARACOMPACTNESS

Theorem 4.1. *Let* B *be any* α*-paracompact subset of* Y . *If* f *is a closed and continuous mapping of a space* X *onto a space* Y *such that for each* $y \in B$, $f^{-1}(y)$ *is* α*-paracompact, then* $f^{-1}(B)$ *is* α*-paracompact.*

Proof. *Let*

$$\mathcal{U} = \{U_i : i \in I\}$$

be any X-open covering of $f^{-1}(B)$. For each point $y \in B$, there exists an X-open X-locally finite family $\mathcal{V}_y$ which refines $\mathcal{U}$ such that

$$f^{-1}(y) \subset \cup\{V_y : V_y \in \mathcal{V}_y\} = V_y^{*}.$$

Since f is closed there exists an open set S_y of Y such that $y \in S_y$ and

$$f^{-1}(y) \subset f^{-1}(S_y) \subset V_y^{*}.$$

Now

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$$S = \{S_y : y \in B\}$$

is a Y-open covering of B. Since B is α-paracompact, there exists a Y-open Y-locally finite family

$$K = \{K_j : j \in J\}$$

which refines S and covers B.
Now,

$$\{f^{-1}(K_j) : j \in J\}$$

is an X-open X-locally finite family which covers $f^{-1}(B)$. For each $j \in J$ there exists a $y_j \in B$ such that

$$K_j \subset S_{y_j}.$$

Hence

$$f^{-1}(K_j) \subset f^{-1}(S_{y_j}) \subset V^{*}_{y_j}.$$

For such y_j let

$$V = \{f^{-1}(K_j) \cap V_{y_j} : j \in J,\ V_{y_j} \in \mathcal{V}_{y_j}\}.$$

The family V is an X-open covering of $f^{-1}(B)$ which refines U. It follows easily that V is X-locally finite, hence $f^{-1}(B)$ is α-paracompact.

Corollary 4.1. If f is a closed and continuous mapping of a space X onto a paracompact space Y such that for each point $y \in Y$ $f^{-1}(y)$ is α-paracompact, then X is paracompact.

Theorem 4.2. *If* F *is an almost closed upper semi-*

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continuous and an open multifunction of an almost paracompact space X onto a space Y such that for each y ∈ Y the set $F^-(y)$ is α-nearly compact, for each point x ∈ X the set F(x) is α-nearly paracompact and for every proper closed subset G of Y the set $F^-(G)$ is a proper subset of X, then Y is almost paracompact.

Proof. *Let*

$$\mathcal{U} = \{U_i : i \in I\}$$

be any regularly open covering of Y. Since for each x ∈ X the set F(x) is α-nearly paracompact and $\mathcal{U}$ is a regularly open covering of Y, there exists a Y-open Y-locally finite family

$$\{G_j : j \in J_x\}$$

which refines $\mathcal{U}$ such that

$$F(x) \subset \cup\{G_j : j \in J_x\}.$$

Let

$$G_x = \cup\{G_j : j \in J_x\}.$$

Thus G_x is an open set such that $F(x) \subset G_x$. Since F is upper semicontinuous, the collection

$$\{F^+(G_x) : x \in X\}$$

is an open covering of X. X is almost paracompact, so there exists an open locally finite family

$$\mathcal{V} = \{V_k : k \in K\}$$

which refines $\{F^+(G_x) : x \in X\}$ such that

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$$X = \overline{\cup \{V_k : k \in K\}} = \cup \{\overline{V}_k : k \in K\}.$$

Next F is almost closed and open such that $F^-(y)$ is α-nearly compact, hence, by Theorem 1.5.

$$F(\mathcal{V}) = \{F(V_k) : k \in K\}$$

is a locally finite and open family.

For each $k \in K$ pick $x_k \in X$ such that

$$V_k \subset F^+(G_{x_k})$$

then $F(V_k) \subset F(F^+(G_{x_k})) \subset G_{x_k}$.
Let

$$\mathcal{V}^* = \{F(V_k) \cap G_j : j \in J_{x_k}, k \in K\}.$$

$\mathcal{V}^*$ is an open locally finite family which refines $\mathcal{U}$.
Now, we have to prove that

$$\bigcup_{k\in K} \bigcup_{j\in J_{x_k}} \overline{F(V_k) \cap G_j} = Y,$$

i.e.

$$\overline{\bigcup_{k\in K} \bigcup_{j\in J_{x_k}} F(V_k) \cap G_j} = \overline{\bigcup_{k\in K} F(V_k) \cap \bigcup_{j\in J_{x_k}} G_j} =$$

$$= \overline{\bigcup_{k\in K} F(V_k)} = \bigcup_{k\in K} \overline{F(V_k)} = Y.$$

Suppose that $\bigcup_{k\in K} \overline{F(V_k)}$ is a proper closed subset of Y.

There exists a point $x \in X$ such that $F(x) \subset Y \setminus \bigcup_{k\in K} \overline{F(V_k)}$. Now,

$$F^+(Y \setminus \bigcup_{k\in K} \overline{F(V_k)}$$

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is a nonempty open set such that

$$F^+(Y \setminus \bigcup_{k\in K} \overline{F(V_k)}) \cap V_{k_j} \neq \emptyset$$

for some $k_j \in K$.

Let u be an element of X, such that

$$u \in F^+(Y \setminus \bigcup_{k\in K} \overline{F(V_k)}) \cap V_{k_j}.$$

From $F(u) \subset Y \setminus \bigcup_{k\in K} \overline{F(V_k)}$ and $F(u) \subset F(V_{k_j})$, we have

$$(Y \setminus \bigcup_{k\in K} \overline{F(V_k)}) \cap F(V_{k_j}) \neq \emptyset .$$

This is a contradiction.

It follows that

$$\bigcup_{k\in K} \overline{F(V_k)} = Y.$$

Hence Y is almost paracompact.

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REZIME

O PRESLIKAVANJU I PARAKOMPAKTNOSTI

U radu se ispituju neke osobine parakompaktnosti, α-regularnih, α-Hausdorffovih podskupova, zatvorenih, skoro zatvorenih, neprekidnih preslikavanja, kao i višeznačno preslikavanje skoro parakompaktnog prostora.

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A PAIR OF COMMUTING MAPPINGS WITH A COMMON FIXED POINT

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ABSTRACT

Two common fixed point theorems for two commuting mappings of a complete metric space into itself are given. These theorems generalize some earlier of Ćirić and the first author.

The first author [3], generalizing a result of Ćirić [1], defines a mapping T of a metric space (X,d) into itself to be a quasi-contraction if

$$(1)\qquad d(T^p x, T^q y) \le c \cdot \max\{d(T^r x, T^s y), d(T^r x, T^{r'} x), d(T^s y, T^{s'} y):$$

$$0 \le r, r' \le p;\ 0 \le s, s' \le q\}$$

for all x,y in X, where $0 \le c < 1$ and p, q are some fixed positive integers.

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The following result holds, see [3].

Theorem 1. *Let T be a continuous quasi-contraction of a complete metric space (X,d) into itself. Then T has a unique fixed point in X.*

An interesting generalization of this theorem was proved by Park and Rhoades [7] using a generalized version of a contractive condition of Hegedüs and Szilágyi [6].

Theorem 1 was also extended in [5] for a pair of continuous and commuting mappings S and T of a bounded, complete metric space into itself.

Here we give a further generalization of Theorem 1 for a pair of commuting mappings without requiring necessarily the simultaneous continuity of both S and T and by replacing the condition that X is bounded by the following condition

$$(2) \qquad \sup\{d(S^{r+i}T^{j}x, S^{r}T^{j'}x), d(T^{r+j}S^{i}x, T^{r}S^{i'}x) : r \geq 0,$$

$$0 \leq i,i' \leq p,\ 0 \leq j,j' < q\} = L < \infty,$$

for some particular x in X, p and q being fixed positive integers.

It is well known in the literature that the common fixed point of two mappings S and T is deduced as a limit of the sequence

$$(3) \qquad \{x, Tx, STx, \ldots, (ST)^{n}x, T(ST)^{n}x, \ldots\}.$$

Usually one first of all shows that the sequence (3) is bounded. As pointed out in [4], inequality (2) is certainly satisfied if X is bounded,but generally (2) does not imply that (3) is bounded if X is unbounded. To see this, consider $X = [0,\infty)$ with the euclidean metric and $Sx = x + 1$, $Tx = x + 2$ for all x in X. Then inequality (2), is satisfied, since $L = p + 2q - 3$, but the sequence (3) is unbounded.

Given a mapping T of (X,d) into itself, we now defi-

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ne the function $D_T(x)$ by

$$D_T(x) = d(x,Tx)$$

for all x in X. Motivated by a paper of Ćirić [2], we say that T has the property (α) in X if for any x_0 in X and for any sequence $\{x_n\}$ converging to x_0, we have

$$\lim_{n\to\infty} \sup D_T(x_n) \geq D_T(x_0).$$

Of course, any continuous mapping T has the property (α).

Theorem 2. *Let* S *and* T *be commuting mappings of a complete metric space* (X,d) *into itself, with* S *or* T *having property* (α), *satisfying the inequality*

$$(4) \qquad d(S^p x, T^q y) \leq c \cdot \max\{d(S^i x, S^{i'} x), d(T^j y, T^{j'} y), d(S^i x, T^j y):$$
$$0 \leq i,i' \leq p; \quad 0 \leq j,j' \leq q\}$$

for all x, y *in* X, *where* $0 \leq c < 1$ *and* p, q *are fixed positive integers. Suppose further that (2) holds for some particular* x *in* X. *Then* S *and* T *have a unique common fixed point* z. *Further,* z *is the unique fixed point of* S *and* T.

Proof. Using inequality (4) we have for $r \geq p$ and $0 \leq j < q$

$$d(S^r T^j x, T^q x) \leq c \cdot \max\ d(S^{r-i} T^j x, S^{r-i'} T^j x), d(T^{j'} x, T^{j''} x),$$
$$d(S^{r-i} T^j x, T^{j'} x) : 0 \leq i,i' \leq p; \ 0 \leq j',j'' \leq q\}$$
$$\leq c \cdot \max\{L, L, d(S^{r-i} T^j x, S^r T^j x) + d(S^r T^j x, T^q x) + d(T^q x, T^{j'} x):$$
$$0 \leq i \leq p; \ 0 \leq j' \leq q\}$$
$$\leq c[2L + d(S^r T^j x, T^q x)]$$

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on using (2). Thus

$$d(S^r T^j x, T^q x) \leq 2Lc/(1 - c)$$

for $r \geq p$ and $0 \leq j < q$. It follows that the set

$$\{S^r T^j x : r \geq 0;\ 0 \leq j < q\}$$

is bounded. We can prove similarly that the set

$$\{T^r S^i x : r \geq 0;\ 0 \leq i < p\}$$

is also bounded and it follows that

$$(5) \qquad \sup\{d(S^r T^j x, S^n T^{j'} x), d(T^r S^i x, T^n S^{i'} x) : r,n \geq 0;$$

$$0 \leq i,i' < p;\ 0 \leq j,j' < q\} = K < \infty.$$

Let us now suppose that the set

$$A = \{S^{n-r} T^r x : n \geq 0;\ 0 \leq r \leq n\}$$

is unbounded. Then there exist integers r and n, with $n-r \geq p$, such that

$$(6) \qquad d(S^{n-r} T^r x, T^q x) > \max\{Kc/(1-c), K\}$$

and

$$(7) \qquad d(S^{n-r} T^r x, T^q x) > \max\{d(S^{m-i} T^i x, T^q x), d(S^{n-j} T^j x, T^q x) :$$

$$0 \leq i \leq m;\ 0 \leq m < n;\ 0 \leq j < r\}.$$

Now choose an integer k such that

$$(8) \qquad d(S^{n-r} T^r x, T^q x) > c^k \cdot \max\{d(S^i T^j x, S^{i'} T^{j'} x) :$$

$$0 \leq i,i' \leq n;\ 0 \leq j,j' < n\}$$

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Using inequality (4) we have

$$d(S^{n-r}T^r x, T^q) \le c \cdot max\{d(S^{n-r-i}T^r x, S^{n-r-i'}T^r x),$$

$$d(T^j x, T^{j'} x), d(S^{n-r-i}T^r x, T^j x) :$$

$$0 \le i, i' \le p; \quad 0 \le j, j' \le q\}$$

$$\le c \cdot max\{d(S^{n-r-i}T^r x, S^{n-r-i'}T^r x), K, d(S^{n-r-i}T^r x, T^q x) +$$

$$+ d(T^q x, T^j x) : 0 \le i, i' \le p; \quad 0 \le j \le q\}$$

$$\le c \cdot max\{d(S^{n-r-i}T^r x, S^{n-r-i'}T^r x),$$

$$d(S^{n-r}T^r x, T^q x) + K : 0 \le i, i' \le p\}$$

because of (5) and (7).

Now

$$d(S^{n-r}T^r x, T^q x) \le c[d(S^{n-r}T^r x, T^q x) + K]$$

implies

$$d(S^{n-r}T^r x, T^q x) \le Kc/(1 - c),$$

contradicting inequality (6). We must therefore have

$$(9) \qquad d(S^{n-r}T^r x, T^q x) \le c \cdot max\{d(S^{n-r-i}T^r x, S^{n-r-i'}T^r x) :$$

$$0 \le i, i' \le p\},$$

where $r \ge q$, otherwise inequality (6) would again be contradicted. We can also omit all terms on the right-hand side of inequality (9) where $n-r-i < p$ and $n-r-i' < p$ because of (5) and inequality (6). Inequality (4) can therefore be applied to the remaining terms on the right-hand side of inequality (9) to give terms of the form

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$$d(S^{i}T^{j}x, S^{i'}T^{j'}x)$$

where $0 \le i,i' \le n-r$ and $0 \le j,j' \le r$. Inequality (4) can be applied to these resulting terms either indefinitely or until terms of the form

(10) $$\{d(S^{i}T^{j}x, S^{i'}T^{j'}x) : 0 \le i,i' \le n-r;\ 0 \le j,j' < q\}$$

or

(11) $$\{d(S^{i}T^{j}x, S^{i'}T^{j'}x) : 0 \le i,i' < p;\ 0 \le j,j' \le r\}$$

or

(12) $$\{d(S^{i}T^{j}x, S^{i'}T^{j'}x) : 0 \le i \le n-r\ ;\ 0 \le i' < p\ ;\ 0 \le j \le r\ ;\ 0 \le j' < q\}$$

are obtained. Terms obtained after k applications of inequality (4) can be omitted because of inequality (8) and terms of the form (10) or (11) can be omitted because of (5). We must therefore have

$$d(S^{n-r}T^{r}x, T^{q}x) \le c \cdot max\{d(S^{i}T^{j}x, S^{i'}T^{j'}x) : 0 \le i \le n-r;\ 0 \le i' < p;\ 0 \le j \le r;\ 0 \le j' < q\}$$

$$\le c \cdot max\{d(S^{i}T^{j}x, T^{q}x) + d(T^{q}x, S^{i'}T^{j'}x) : 0 \le i \le n-r;\ 0 \le i' < p;\ 0 \le j \le r;\ 0 \le j' < q\}$$

$$\le c[d(S^{n-r}T^{r}x, T^{q}x) + K]$$

because of (5) and (7), again leading to a contradiction of inequality (6).

The set A therefore must be bounded and so

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$$\sup\{d(S^nT^rx,S^{n'}T^{r'}x) : n,n',r,r' = 0,1,2,\ldots\} = M<\infty.$$

Without loss of generality, we will now suppose that $p \geq q$ and claim that

$$d((ST)^{np}u,(ST)^{np}v) \leq c^n \cdot \max\{d(S^iT^ju,S^{i'}T^{j'}u),$$

$$d(S^iT^ju,S^{i'}T^{j'}v),d(S^iT^jv,S^{i'}T^{j'}v) :$$

(13)

$$0 \leq i,i',j,j' \leq np\}$$

for $n = 0,1,2,\ldots$ and all u, v in X. This is certainly true when $n = 0$. Assume true for some n. Then

$$d((ST)^{(n+1)p}u,(ST)^{(n+1)p}v) =$$

$$= d((ST)^{np}(ST)^pu,(ST)^{np}(ST)^pv)$$

$$\leq c^n.\max\{d(S^iT^j(ST)^pu,S^{i'}T^{j'}(ST)^pu),$$

$$d(S^iT^j(ST)^pu,S^{i'}T^{j'}(ST)^pv),d(S^iT^j(ST)^pv,S^{i'}T^{j'}(ST)^pv$$

$$0 \leq i,i',j,j' \leq np\}$$

$$\leq c^{n+1}.\max\{d(S^iT^ju,S^{i'}T^{j'}u),d(S^iT^ju,S^{i'}T^{j'}v),$$

$$d(S^iT^jv,S^{i'}T^{j'}v) : 0 \leq i,i',j,j' \leq (n+1)p\},$$

on using inequality (4). Inequality (13) follows by unduction.

Putting $u = (ST)^ix$ and $v = T(ST)^ix$ in inequality (13), we have

$$d((ST)^{np+i}x,T(ST)^{np+i}x) \leq c^n \max\{d(S^iT^jx,S^{i'}T^{j'}x) :$$

$$0 \leq i,i',j,j' \leq (n+1)p\}$$

$$\leq c^nM$$

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for $i = 1,2,\ldots,p$ and $n = 0,1,2,\ldots$. We can prove similarly

$$d(T(ST)^{np+i-1}x,(ST)^{np+i}x) \leq c^n M$$

for $i = 1,2,\ldots,p$ and $n = 0,1,2,\ldots$. It follows that the sequence (3) is a Cauchy sequence in the complete metric space X and so has a limit z in X.

Now suppose that T has the property (α). Then we have

$$\begin{aligned} \lim_{n\to\infty} \sup D_T((ST)^n x) &= \lim_{n\to\infty} \sup d((ST)^n x, T(ST)^n x) \\ &= d(z,z) = 0 \\ &\geq D_T(z) = d(z,Tz) \end{aligned}$$

and hence $Tz = z$.

Putting $u = S^i x$ and $v = x$ in inequality (13), it follows that

$$d((ST)^{np}S^i x,(ST)^{np}x) \leq c^n M$$

for $i = 0,1,\ldots,p$ and so the sequence

$$\{(ST)^{np}S^i x : n = 1,2,\ldots\} \tag{14}$$

also converges to z for $i = 0,1,\ldots,p$.

Further,

$$\begin{aligned} d((ST)^{np}S^p x,Sz) &= d(S^p(ST)^{np}x,ST^q z) = d(S^p(ST)^{np}x,T^q Sz) \\ &\leq c\cdot max\{d((ST)^{np}S^i x,(ST)^{np}S^{i'}x),d(T^j Sz,T^{j'}Sz), \\ &\quad d((ST)^{np}S^i x,T^j Sz) : 0 \leq i,i' \leq p;\ 0 \leq j,j' \leq q\} \\ &= c\cdot max\{d((ST)^{np}S^i x,(ST)^{np}S^{i'}x),d(Sz,Sz), \\ &\quad d((ST)^{np}S^i x,Sz) : 0 \leq i,i' \leq p\}, \end{aligned}$$

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since T commutes with S. Letting n tend to infinity, it follows that

$$d(z,Sz) \leq cd(z,Sz)$$

and so z is also a fixed point of S. A similar proof can of course be given if one assumes that S has the property (α) instead of T. The uniqueness of z follows easily. This completes the proof of the theorem.

Remark 1. The example given [4] shows that the commutativity of S and T is necessary in Theorem 2.

Remark 2. Note that Theorem 2 is false if neither S nor T have the property (α). Indeed, let X = [0,1] with the euclidean metric, let S0 = 1/2, Sx = x/4 if x ≠ 0 and let T0 = 1, Tx = x/2 if x ≠ 0. Then S and T commute and inequality (2) holds since X is bounded. Further, an easy calculation shows that inequality (4) is satisfied with c = 1/4, p = 2 and q = 4. Neither S nor T has the property (α) because for any sequence $\{x_n\}$, with $x_n \neq 0$ for any integer n, converging to 0

$$\lim_{n\to\infty} \sup D_S(x_n) = \lim_{n\to\infty} \sup \frac{3}{4} x_n = 0 < \frac{1}{2} = D_S(0)$$

and

$$\lim_{n\to\infty} \sup D_T(x_n) = \lim_{n\to\infty} \sup \frac{1}{2} x_n = 0 < 1 = D_T(0).$$

In the particular case that either p = 1 or q = 1, the condition that S or T has the property (α) is not necessary in Theorem 2. Indeed the following result holds.

Theorem 3. *Let S and T be commuting mappings of a complete metric space* (X,d) *into itself satisfying the inequality*

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$$d(S^p x, Ty) \le c \cdot \max\{d(S^i x, S^j x), d(y, Ty), d(S^i x, y),$$

(15)

$$d(S^i x, Ty) : 0 \le i, j \le p\}$$

for all x, y *in* X, *where* $0 \le c < 1$ *and* p *is a fixed positive integer. Suppose further that for some particular* x *in* X,

$$\sup\{d(S^{r+i}x, S^r x), d(T^r S^i x, T^r S^j x) :$$

(16)

$$r \ge 0;\ 0 \le i, j < p\} < \infty \quad .$$

Then S *and* T *have a unique common fixed point* z. *Further,* z *is the unique fixed point of* S *and* T.

Proof. It follows as in the proof of Theorem 2 that the sequence (14) converges to a point z in X for i = 0,1,...,p. Then using inequality (15) we have

$$d((ST)^{np}S^p x, Tz) \le c \cdot \max\{d((ST)^{np}S^i x, (ST)^{np}S^i x), d(z, Tz),$$

$$d((ST)^{np}S^i x, z), d((ST)^{np}S^i x, Tz):$$

$$0 \le i, j \le p\}$$

and letting n tend to infinity it follows that

$$d(z, Tz) \le cd(z, Tz).$$

Thus Tz = z and we then have Sz = z as in the proof of Theorem 2.

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REZIME

PAR KOMUTIRAJUĆIH PRESLIKAVANJA SA ZAJEDNIČKOM NEPOKRETNOM TAČKOM

Dve teoreme o zajedničkoj nepokretnoj tački su dokazane za komutirajuća preslikavanja kompletnog metričkog prostora u sebe. Ove teoreme uopštavaju ranije rezultate Ćirića i prvog autora.

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ON A GENERALIZATION OF A CRAMER THEOREM*

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ABSTRACT

The paper extends the well-known theorem of Cramer concerning sufficient conditions for the stochastic process to have multiplicity equal to one.

☆

Let the stochastic process $x(t)$ be given by the Cramer representation:

$$x(t) = \int_a^t g(t,u)dz(u), \tag{1}$$

$u \leqslant t$, $u,t \in (a,b) = T$, where $z(u)$ is a process of orthogonal increments such that:

$$Ez(u) = 0, \quad \text{and} \quad Ez^2(u) = F(u),$$

and $g(t,u)$, $u \leqslant t$, is a nonrandom function from space $L^2(dF(u))$:

$$\int_a^t g^2(t,u)d\bar{F}(u) < \infty.$$

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The second order process $x(t)$, $t \in T$ is continuous to the left and purely nondeterministic.

Let us introduce the next conditions for $g(t,u)$ and $z(u)$:

R_1 The functions $g(t,u)$ and $g'_t(t,u)$ are bounded and continuous for $u \leqslant t$, $u,t \in T$.

R_2 $g(t,t) = 1$ for all $t \in T$.

R_3 The function $F(u) = Ez(u)$ is absolutely continuous and not identically constant and $f(u) = F'(u)$ has at most a finite number of discontinuity points in any finite subinterval of T.

Cramer in [2] proved the following statement:

Theorem. *The process* $x(t)$, $t \in T$ *from (1) which satisfies the conditions* R_1, R_2, R_3 *has a multiplicity one.*

Condition R_2 *and* $g(t,t) > 0$ *for all* $t \in T$, *are equivalent.*

It is easy to see that R_2 implies $g(t,t) > 0$ for all $t \in T$. Since $g(t,t) > 0$ for all $t \in T$, we may introduce the transformations:

$$\bar{g}(t,u) = g(t,u)/g(u,u),$$

$$\overline{dz}(u) = g(u,u) \cdot dz(u),$$

$u \leqslant t$, $u,t \in T$, such that R_2 holds for the process $x(t)$:

$$\bar{x}(t) = \int_a^t \bar{g}(t,u) \cdot \overline{dz}(u).$$

Remark. When $g(t,t) < 0$ for all $t \in T$, we may introduce the same transformations.

Cramer also proved that the upper conditions of the preceding theorem imply $x(t)$, $t \in T$ is given by a purely canonical representation (in Hida-Cramer mean), see theorem 5.2. in

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[2]. But it is not valid. The following example shows it.

Example. Let $x(t)$ be given by the expression:

$$x(t) = \int_0^t (-3t + 4u)dz(u),$$

$u \leqslant t$, $u,t \in (0,\tau) = T$, where $f(u) = F'(u) = 1$. Then, there exists a magnitude y:

$$y = \int_0^\tau u^2 dz(u),$$

from the Hilbert space $H(x)$ generated by $x(t)$, $t \in T$, such that the scalar product:

$$(x(t),y) = Ex(t)\cdot y = 0.$$

That means that the family $g(t,u) = -3t + 4u$ satisfying R_1 and R_2 is not complete in space $L^2(du)$, or $x(t)$, $t \in T$ is not given by the Cramer representation.

If we omit condition R_2 from the preceding theorem, this gives us the main result.

Theorem. *The process* $x(t)$, $t \in T$ *represented by (1) which satisfies the conditions* R_1 *and* R_2 *has multiplicity* $N = 1$.

Proof. Let us suppose that $N > 1$. For example let N be two. By condition R_1 the function $g(t,t) = (g_1(t,t),g_2(t,t))$, $t \in T$ may be equal to zero in a finite number of isolated points or on a finite number of subintervals of positive measures. Let us denote the union of intervals where $g_n(t,t) = 0$ by A_n and the union of intervals where $g_n(t,t) < 0$ by B_n, $n = 1,2$. From condition R_3 there exists a finite subinterval $T_1 \subset T$, $T_1 = (a_1,b_1)$ such that the derivatives $f_1(u)$ and $f_2(u)$ are continuous and not equal to zero. We choose such T_1 but we suppose, moreover, that T_1 does not contain the isolated zeros of the functions $g_1(t,t)$ and $g_2(t,t)$, and that T_1 is contained in

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$\bigcap_{n=1}^{2} C_n$ where C_n is one of the next sets: A_n, B_n, $T \setminus (A_n \cup B_n)$, for $n = 1,2$. For instance, let us take $T_1 \subset A_1 \cap (T \setminus (A_2 \cup B_2))$. Then, on the interval T_1 : $g_1(t,t) = 0$ and $g_2(t,t) > 0$, for all $t \in T$.

Let t be any point in T_1 and let $h(u)$ be a function in $L^2(dF_1(u) \times dF_2(u))$ such that the next relation is valid:

$$\sum_{n=1}^{2} \int_{a_1}^{s} h_n(u) g_n(s,u) f_n(u) du = 0,$$

$s \leqslant t$, $u \leqslant s$, $t \in T$. By condition R_1 the relation may be differentiated with respect to s, and we obtain:

$$\sum_{n=1}^{2} \int_{a_1}^{s} g'_{n_s}(s,u)\, h_n(u)\, f_n(u)\, du + \sum_{n=1}^{2} g_n(s,s)\, h_n(s)\, f_n(s) = 0,$$

for $s \in (a_1,t]$. This implies that:

$$\int_{a_1}^{s} g'_{1_s}(s,u) h_1(u) f_1(u) du + \int_{a_1}^{s} g'_{2_s}(s,u) h_2(u) f_2(u) du +$$

$$+ g_2(s,s) h_2(s) f_2(s) = 0,$$

for $s \in (a_1,t]$. This equation is satisfied, if for example:

$$\int_{a}^{s} g'_{1_s}(s,u) h_1(u) f_1(u) du = 1,$$

and

$$\int_{a}^{s} g'_{2_s}(s,u) h_2(u) f_2(u) du + g_2(s,s) h_2(s) f_2(s) = -1.$$

These are nonhomogeneous integral equations of Volterra of the first and second type, and for each there exist bounded, continuous and not almost everywhere equal to zero solutions $f_1(s)h_1(s)$ and $f_2(s)h_2(s)$ for $s \in (a_1,t]$. By our hypothesis

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about $f_1(u)$ and $f_2(u)$ on T_1, it follows that:

$$\int_a^t h_1^2(u)dF_1(u) + \int_a^t h_2^2(u)dF_2(u) > 0. \tag{2}$$

That means $g(t,u) = (g_1(t,u), g_2(t,u))$ is not complete in $L^2(dF_1(u) \times dF_2(u))$.

We can show in a similar way that (2) holds when T_1 is contained in one of the remaining eight sets.

When we suppose that N is equal to any natural number $N > 2$, the conclusion is the same as before.

The proof is completed.

The representation (1) of $x(t)$, $t \in T$ which satisfies the conditions R_1 and R_2 can be not purely canonical.

Example. Let the process $x(t)$ be given by:

$$x(t) = \int_0^t \left(\frac{3}{5} - 3\frac{u}{t} + 5\frac{u^2}{t^2} - 3\frac{u^3}{t^3} + \frac{2}{5}\frac{u^5}{t^5}\right)dz(u),$$

where $u \leq t$, $u,t \in (0,\tau) = T$ and $f(u) = F'(u) = 1$.
The functions:

$$g(t,u) = \frac{3}{5} - 3\frac{u}{t} + 5\frac{u^2}{t^2} - 3\frac{u^3}{t^3} + \frac{2}{5}\frac{u^5}{t^5}$$

$$g'_t(t,u) = \frac{1}{t}\left(3\frac{u}{t} - 10\frac{u^2}{t^2} + 9\frac{u^3}{t^3} - 2\frac{u^5}{t^5}\right),$$

for $u \leq t$ are bounded and continuous on TxT. From the fact: $g(t,t) = g'_t(t,t) = 0$ for all $t \in T$, follows the limitation of $g(t,u)$ and $g'_t(t,u)$ when $t \to 0$. The process $x(t)$, $t \in T$ has multiplicity equal to one. But this representation of $x(t)$, $t \in T$ is not purely canonical. There exists the magnitude $y \in H(x)$:

$$y = \int_0^\tau u^2 dz(u),$$

such that the following relation is valid:

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$$\int_0^t \left(\frac{3}{5} - 3\frac{u}{t} + 5\frac{u^2}{t^2} - 3\frac{u^3}{t^3} + \frac{2}{5}\frac{u^5}{t^5}\right)\cdot u^2 du = 0.$$

Hence, the conditions R_1 and R_3 do not imply that the process $x(t)$, $t \in T$ is given by the Cramer representation (or purely canonical one).

The problem is now if we can omit the condition of a purely canonical representation of $x(t)$, $t \in T$ in the proofs of the preceding theorems?

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REZIME

JEDNO UOPŠTENJE KRAMEROVE TEOREME

Rad proširuje dobro poznatu teoremu Cramera koja se odnosi na dovoljne uslove da bi stohastički proces imao multiplicitet jednak jedan.

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A THEOREM ON PARTIAL MIDDLING FOR FUNCTIONAL-DIFFERENTIAL EQUATIONS OF THE NEUTRAL TYPE

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ABSTRACT

In this paper a generalization of Plotnikov's result from [5] is obtained for functional-differential equations of the form: $\dot{x}(t) \in F(t,x_t,\dot{x}_t)$, where F is a multifunction with values which are nonempty compact convex subsets of n-dimensional space R^n.

1. INTRODUCTION

In classical system of functional-differential equations it is possible to middle both complete and partial equations. Complete middling was presented by Bogolubov ([1]).

In this paper, we use a partial middling method in the case of generalized functional-differential equations of the form

$$\dot{x}(t) \in F(t,x_t,\dot{x}_t)$$

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where F is a multifunction with values that are nonempty compact convex subsets of n-dimensional Euclidean space R^n. The application of this method leads to a reduced form of the initial equations system and is useful in the case when the means of certain functions do not exist.

The results of this paper generalize the results of W. Plotnikov ([5]), where the generalized system $\dot{x}(t) \in F(t,x)$ was investigated.

Let C_α and $\mathcal{L}_\alpha$, $\alpha \geq 0$ denote the Banach spaces of all continuous and Lebesgue integrable functions, respectively, of $[-r,\alpha]$ into R^n with the norms

$$\|x\|_\alpha = \sup_{-r\leq t\leq\alpha} |x(t)| \quad \text{and} \quad |y|_\alpha = \int_{-r}^{\alpha} |y(t)|dt$$

for $x \in C_\alpha$ and $y \in \mathcal{L}_\alpha$ respectively, where $|\cdot|$ denotes the Euclidean norm. For a given function $u : [-r,T] \to R^n$ and fixed $t \in [0,T]$, we denote $u_t(s) = u(t + s)$ for $s \in [-r,0]$, $r \geq 0, T > 0$. Finally, let us denote by (Comp R^n,H) and (Conv R^n,H) the metric spaces of all nonempty compact and nonempty compact and convex, respectively, subsets of n-dimensional Euclidean space R^n with the Hausdorff metric H.

2. THE THEOREM ON PARTIAL MIDDLING

Let $F^i : [0,\infty) \times C_o \times \mathcal{L}_o \to \text{Conv } R^n$ $(i = 1,2)$ satisfy the following conditions:

(a) $F^i(\cdot,u,v) : [0,\infty) \to \text{Conv } R^n$ is measurable for fixed $(u,v) \in C_o \times \mathcal{L}_o$,

(b) there exists a $M > 0$ such that $H(F^i(t,u,v),\{0\}) \leq M$ for $(t,u,v) \in [0,\infty) \times C_o \times \mathcal{L}_o$,

(c) $F^i(t,\cdot,\cdot) : C_o \times \mathcal{L}_o \to \text{Conv } R^n$ satisfies for fixed $t \in [0,\infty)$ the Lipschitz condition of the form $H(F^i(t,u,v),F^i(t,\bar{u},\bar{v})) \leq k[\|u - \bar{u}\|_o + |v - \bar{v}|_o]$ where $k > 0$, $u,\bar{u} \in C_o$ and $v,\bar{v} \in \mathcal{L}_o$,

(d) there exists a limit

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$$\lim_{L\to\infty} \frac{1}{L} H\Big(\int_0^L F^1(t,u,v)dt,\ \int_0^L F^2(t,u,v)dt\Big) = 0$$

uniformly with respect to $(u,v) \in C_0 \times \mathcal{L}_0$.

In this part we shall study differential inclusions of the form

(1) $$\dot{x}^1(t) \in \varepsilon F^1(t,x_t^1,\dot{x}_t^1) \quad \text{for a.e. } t \geq 0$$

and

(2) $$\dot{x}^2(t) \in \varepsilon F^2(t,x_t^2,\dot{x}_t^2) \quad \text{for a.e. } t \geq 0$$

where $\varepsilon > 0$ is a small parameter. We shall consider (1) and (2) together with the initial conditions

(3) $$x^1(t) = x^2(t) = \varphi(t) \quad \text{for } t \in [-r,0]$$

where $\varphi : [-r,0] \to R^n$ is a given absolutely continuous function.

In paper [4] the following theorem has been proved.

Theorem 1. *Let* $\delta : [0,T] \to R$ *be a nonnegative Lebesgue integrable function and* $\varphi \in C_0$ *be absolutely continuous. Suppose* $F : [0,T] \times C_0 \times \mathcal{L}_0 \to \mathrm{Comp}\, R^n$ *satisfies (a) (b) and (c) of the form*

$$H(F(t,u,v),\ F(t,\bar{u},\bar{v})) \leq k(t)[\|u - \bar{u}\|_0 + |v - \bar{v}|_0]$$

where $k : [0,T] \to R^+$ *is a Lebesgue integrable function,* $u,\bar{u} \in C_0$ *and* $v,\bar{v} \in \mathcal{L}_0$. *Furthermore let* $y : [-r,T] \to R^n$ *be an absolutely continuous mapping such that*

(e) $$y(t) = \varphi(t) \quad \textit{for } t \in [-r,0],$$

(f) $$d(\dot{y}(t),\ F(t,y_t,\dot{y}_t)) \leq \delta(t) \quad \textit{for a.e. } t \in [0,T].$$

Then there is a solution $x(\cdot)$ *of an initial-value problem*

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$$\begin{cases} \dot{x}(t) \in F(t,x_t,\dot{x}_t) & \text{for a.e. } t \in [0,T] \\ x(t) = \varphi(t) & \text{for } t \in [-r,0] \end{cases}$$

such that

$$|x(t) - y(t)| \leq \xi(t) \text{ for } t \in [0,T]$$

and

$$|\dot{x}(t) - \dot{y}(t)| \leq \delta(t) + 2k(t)\xi(t) \text{ for a.e. } t \in [0,T]$$

where

$$\xi(t) = \int_0^t \delta(s)\, e^{2[m(t)-m(s)]}\, ds \text{ and } m(t) = \int_0^t k(r)\,dr.$$

Now we can prove the main result of this paper.

Theorem 2. *Suppose* $F^i : [0,\infty) \times C_0 \times \mathcal{L}_0 \to \operatorname{Conv} R^n$, $(i = 1,2)$ *satisfy the conditions (a) - (d). Then, for each* $\eta > 0$ *and* $T > 0$ *there exists a* $\varepsilon^0(\eta,T) > 0$ *such that for every* $\varepsilon \in (0,\varepsilon^0]$ *the following conditions are satisfied:*

(i) *for each solution* $x^1(\cdot)$ *of (1) there exists a solution* $x^2(\cdot)$ *of (2) such that*

$$|x^1(t) - x^2(t)| \leq \eta \text{ for } t \in [-r,T/\varepsilon] \tag{4}$$

(ii) *for each solution* $x^2(\cdot)$ *of (2) there exists a solution* $x^1(\cdot)$ *of (1) such that (4) holds.*

Proof. Let $x^1(\cdot)$ be a solution of (1) on $[-r,\infty)$. To prove this theorem we shall consider the solution $x^2(\cdot)$ of the inclusion (2) in such a way that for $t \in [-r,0]$, $x^1(t) = x^2(t) = \varphi(t) = \text{const.}$, hence $|x^1(t) - x^2(t)| = 0 < \eta$ and that $t \in [0,T/\varepsilon]$ inequality (4) was satisfied too.

To do this divide the interval $[0,T/\varepsilon]$ on m-subintervals $[t_i,t_{i+1}]$, $i = 0,1,2,\ldots,m-1$, and write a solution $x^1(\cdot)$ in the form:

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$$(5)\qquad \begin{cases} x^1(t) = \varphi(t) = \text{const. for} \quad t \in [-r,0], \\ x^1(t) = x^1(t_i) + \varepsilon \int_{t_i}^{t} v^1(\tau)d\tau \text{ for } t \in [t_i,t_{i+1}] \\ \text{where } t_i = iT/\varepsilon m,\ v^1(t) \in F^1(t,x^1_t,\dot{x}^1_t). \end{cases}$$

Let us consider a function $y^1(\cdot)$ defined by

$$(6)\qquad \begin{cases} y^1(t) = \varphi(t) = \text{const. for } t \in [-r,0], \\ y^1(t) = y^1(t_i) + \varepsilon \int_{t_i}^{t} z^1_{i+1}(\tau)d\tau \quad \text{for } t \in [t_i,t_{i+1}] \end{cases}$$

where $z^1_{i+1}(\cdot)$, $i = 0,1,\ldots,m-1$, are measurable functions such that

$$z^1_{i+1}(t) \in F^1(t,y^1_{t_i},\dot{y}^1_{t_i})$$

and

$$|v^1(t) - z^1_{i+1}(t)| = d(v^1(t),F^1(t,y^1_{t_i},\dot{y}^1_{t_i})) =$$

$$\min_{z(t)\in F^1(t,y^1_{t_i},\dot{y}^1_{t_i})} |v_1(t) - z(t)|.$$

Such measurable mappings z^1_{i+1} exist because set-valued function $F^1(\cdot,y_{t_i},\dot{y}^1_{t_i})$ is measurable and has compact and convex values ([3]).

By virtue of (5), for every $t \in [t_i,t_{i+1}]$ we have

$$|x^1(t) - y^1(t_i)| \le$$

$$(7)\qquad \begin{aligned} &\le |x^1(t) - x^1(t_i)| + |x^1(t_i) - y^1(t_i)| = \\ &= \left|\varepsilon \int_{t_i}^{t} v^1(\tau)d\tau\right| + |x^1(t_i) - y^1(t_i)| \le \\ &\le \varepsilon M(t - t_i) + \delta_i, \end{aligned}$$

where $\delta_i = |x^1(t_i) - y^1(t_i)|$, $i = 0,1,\ldots,m-1$.

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Furthermore for $t \in [t_i, t_{i+1}]$

$$(8) \qquad |v^1(t) - z^1_{i+1}(t)| \leq$$

$$\leq H(F^1(t, x^1_t, \dot{x}^1_t), F^1(t, y^1_{t_i}, \dot{y}^1_{t_i})) \leq$$

$$\leq k[\|x^1_t - y^1_{t_i}\|_0 + |\dot{x}^1_{t_i} - \dot{y}^1_{t_i}|_0].$$

But

$$\|x^1_t - y^1_{t_i}\|_0 \leq \|x^1_t - x^1_{t_i}\|_0 + \|x^1_{t_i} - y^1_{t_i}\|_0 =$$

$$= \sup_{-r \leq s \leq 0} |x^1(t + s) - x^1(t_i + s)| +$$

$$+ \sup_{-r \leq s \leq 0} |x^1(t_i + s) - y^1(t_i + s)|.$$

By the absolutely continuous function $x^1(\cdot)$ it follows that for $t \in [t_i, t_{i+1}]$, $x^1(\cdot)$ is uniformly continuous. Therefore, for every given above $\varepsilon > 0$ there exists $\delta(\varepsilon) > 0$ such that $|x^1(t + s) - x^1(t_i + s)| < \varepsilon$ for each $t \in [t_i, t_{i+1}]$ whenever $|t - t_i| \leq (T/\varepsilon m) < \delta(\varepsilon)$.

Hence

$$\sup_{-r \leq s \leq 0} |x^1(t + s) - x^1(t_i + s)| < \varepsilon .$$

Furthermore

$$\sup_{-r \leq s \leq 0} |x^1(t_i + s) - y^1(t_i + s)| =$$

$$= \sup_{t_i - r \leq \tau \leq t_i} |x^1(\tau) - y^1(\tau)|.$$

On the ground of (5) and (6), we have

$$|x^1(\tau) - y^1(\tau)| \leq |x^1(t_i) - y^1(t_i)| +$$

$$+ \varepsilon \int_{\tau}^{t_i} |v^1(s) - z^1_{i+1}(s)| ds \leq$$

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$$\leq \delta_i + 2\varepsilon M(t_i - \tau) \leq \delta_i + 2\varepsilon Mr.$$

Hence

$$\sup_{-r\leq s\leq 0} |x^1(t_i + s) - y^1(t_i + s)| \leq \delta_i + 2\varepsilon Mr.$$

Then

$$\|x^1_t - y^1_{t_i}\|_0 \leq \varepsilon + \delta_i + 2\varepsilon Mr. \tag{9}$$

By the definition of the norm $|\cdot|_0$ and the definitions of (5) and (6), we obtain

$$|\dot{x}^1_t - \dot{y}^1_{t_i}|_0 = \int_{-r}^{0} |\dot{x}^1(t + s) - \dot{y}^1(t_i + s)|\,ds =$$

$$= \begin{cases} 0 \text{ for every } (t + s),(t_i + s) \in [-r,0], \\ \varepsilon \int_{-r}^{0} |v^1(t + s) - z^1_{i+1}(t_i + s)|\,ds \leq 2\varepsilon Mr \\ \quad \text{for every } (t+s),(t_i+s) \in [0,T/\varepsilon], \\ \int_{-r}^{0} |\varepsilon v^1(t + s) - \dot{\varphi}(t_i + s)|\,ds \leq \varepsilon Mr \text{ for every} \\ \quad (t + s) \in [0,T/\varepsilon] \text{ and } (t_i + s) \in [-r,0]. \end{cases}$$

Hence

$$|x^1_t - y^1_{t_i}|_0 \leq 2\varepsilon Mr\ . \tag{10}$$

Therefore inequality (8) for $t \in [t_i,t_{i+1}]$ can be written down in the form

$$|v^1(t) - z^1_{i+1}(t)| \leq k(\varepsilon + \delta_i + 4\varepsilon Mr)\ . \tag{11}$$

By virtue of (5), (6) and (11), it follows

$$\delta_i = |x^1(t_i) - y^1(t_i)| \leq |x^1(t_{i-1}) - y^1(t_{i-1})| +$$

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$$+ \varepsilon \int_{t_{i-1}}^{t_i} |v^1(\tau) - z^1(\tau)|d\tau \leq$$

$$\leq \delta_{i-1} + k\varepsilon(t_i - t_{i-1})(\varepsilon + \delta_{i-1} + 4\varepsilon Mr) =$$

$$= \delta_{i-1} + \frac{kT}{m}(\varepsilon + \delta_{i-1} + 4\varepsilon Mr) =$$

$$= \delta_{i-1}(1 + \frac{kT}{m}) + \frac{\varepsilon kT}{m}(1 + 4Mr) =$$

$$= \delta_{i-1}(1 + \frac{a}{m}) + \frac{b}{m},$$

where $a = kT$ and $b = \varepsilon kT(1 + 4Mr)$.

Hence

$$\delta_i \leq (1 + \frac{a}{m})[(1 + \frac{a}{m})\delta_{i-2} + \frac{b}{m}] + \frac{b}{m}$$

$$\leq (1 + \frac{a}{m})^i \delta_o + (1 + \frac{a}{m})^{i-1} \frac{b}{m} + \ldots + \frac{b}{m} =$$

$$(12) \qquad = \frac{b}{m}(1 + (1 + \frac{a}{m}) + \ldots + (1 + \frac{a}{m})^{i-1}) =$$

$$= \frac{b}{a}[(1 + \frac{a}{m})^i - 1] \leq \frac{b}{a}(e^a - 1) =$$

$$= \varepsilon(1 + 4Mr)(e^{kT} - 1),$$

where $i = 0,1,\ldots,m-1$.

For $t \in [t_i, t_{i+1}]$ we have

$$(13) \qquad |x^1(t) - x^1(t_i)| = |x^1(t_i) + \varepsilon \int_{t_i}^{t} v^1(\tau)d\tau - x^1(t_i)| \leq$$

$$\leq \varepsilon M(t - t_i) \leq \frac{MT}{m}$$

and

$$(14) \qquad |y^1(t) - y^1(t_i)| \leq \frac{MT}{m} .$$

On the ground of (12), (13) and (14), we obtain

$$(15) \qquad |x^1(t) - y^1(t)| \leq |x^1(t) - x^1(t_i)| +$$

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$$+ |x^1(t_i) - y^1(t_i)| + |y^1(t_i) - y^1(t)|$$

$$\leq \frac{2MT}{m} + \varepsilon(1 + 4Mr)(e^{kT} - 1).$$

Now we shall consider the function

$$(16)\quad \begin{cases} y^2(t) = \varphi(t) = \text{const. for } t \in [-r,0], \\ y^2(t) = y^2(t_i) + \varepsilon \int_{t_i}^{t} z^2_{i+1}(\tau)d\tau \text{ for } t \in [t_i,t_{i+1}], \\ \qquad\qquad\qquad\qquad i = 0,1,\ldots,m-1, \end{cases}$$

where $z^2_{i+1}(\cdot)$, $i = 0,1,\ldots,m-1$, are measurable functions, such that $z^2_{i+1}(t) \in F^2(t,y^1_{t_i},\dot{y}^1_{t_i})$ and $|z^1_{i+1}(t) - z^2_{i+1}(t)| = d(z^1_{i+1}(t),F^2(t,y^1_{t_i},\dot{y}^1_{t_i})) = \min_{z(t)\in F^2(t,y^1_{t_i},\dot{y}^1_{t_i})} |z^1_{i+1}(t) - z(t)|$.

Let us notice that by virtue of condition (d) for each $\eta_1 > 0$ and $T > 0$ there exists a $\varepsilon^0(\eta_1,T) > 0$ such that for every $\varepsilon \leq \varepsilon^0$, we have inequalities

$$(17)\quad H\left(\frac{\varepsilon m}{iT}\int_0^{iT/\varepsilon m} F^1(t,y^1_{t_i},\dot{y}^1_{t_i})dt,\ \frac{\varepsilon m}{iT}\int_0^{iT/\varepsilon m} F^2(t,y^1_{t_i},\dot{y}^1_{t_i})dt\right) \leq$$

$$\leq \frac{\eta_1}{2i}$$

and

$$(18)\quad H\left(\frac{\varepsilon m}{(i+1)T}\int_0^{\frac{(i+1)T}{\varepsilon m}} F^1(t,y^1_{t_i},\dot{y}^1_{t_i})dt,\ \frac{\varepsilon m}{(i+1)T}\int_0^{\frac{(i+1)T}{\varepsilon m}} F^2(t,y^1_{t_i},\dot{y}^1_{t_i})dt\right)$$

$$\leq \frac{\eta_1}{2(1+1)}$$

where $i = 1,2,\ldots,m-1$.

By virtue of (17), (18) and the Hausdorff metric condition (see Lemma 3, ([2])), we have

$$H\left(\int_{\frac{iT}{\varepsilon m}}^{\frac{(i+1)T}{\varepsilon m}} F^1(t,y^1_{t_i},\dot{y}^1_{t_i})dt,\ \int_{\frac{iT}{\varepsilon m}}^{\frac{(i+1)T}{\varepsilon m}} F^2(t,y^1_{t_i},\dot{y}^1_{t_i})dt\right) \leq$$

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$$\leq H\Big(\int_0^{\frac{iT}{\varepsilon m}} F^1(t,y^1_{t_i},\dot{y}^1_{t_i})dt,\ \int_0^{\frac{iT}{\varepsilon m}} F^2(t,y^1_{t_i},\dot{y}^1_{t_i})dt\Big) +$$

$$+ H\Big(\int_0^{\frac{(i+1)T}{\varepsilon m}} F^1(t,y^1_{t_i},\dot{y}^1_{t_i})dt,\ \int_0^{\frac{(i+1)T}{\varepsilon m}} F^2(t,y^1_{t_i},\dot{y}^1_{t_i})dt\Big) \leq$$

$$\leq \frac{\eta_1 T}{\varepsilon m}$$

Then

$$\Big|\int_{\frac{iT}{\varepsilon m}}^{\frac{(i+1)T}{\varepsilon m}} (z^1_{i+1}(\tau) - z^2_{i+1}(\tau))d\tau\Big| \leq \frac{\eta_1 T}{\varepsilon m}$$

and

$$|y^1(t_{i+1}) - y^2(t_{i+1})| \leq |y^1(t_i) - y^2(t_i)| +$$

$$+ \Big|\varepsilon\int_{t_i}^{t_{i+1}} (z^1_{i+1}(\tau) - z^2_{i+1}(\tau))d\tau\Big| \leq \tag{19}$$

$$\leq |y^1(t_i) - y^2(t_i)| + \frac{\eta_1 T}{m} \leq \ldots \leq$$

$$\leq \frac{m\eta_1 T}{m} = \eta_1 T, \quad \text{where } i = 0,1,\ldots,m-1.$$

Using the inequality (14), (19) and the fact that for $t \in [t_i,t_{i+1}]$, $|y^2(t) - y^2(t_i)| \leq \frac{MT}{m}$ we have

$$|y^1(t) - y^2(t)| \leq |y^1(t) - y^1(t_i)| +$$

$$+ |y^1(t_i) - y^2(t_i)| + |y^2(t_i) - y^2(t)| \tag{20}$$

$$\leq \frac{2MT}{m} + \eta_1 T .$$

By assumption (c) it follows that

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$$H(F^2(t,y^2_t,\dot{y}^2_t),F^2(t,y^1_{t_i},\dot{y}^1_{t_i})) \leq$$

$$k[\|y^2_t - y^1_{t_i}\|_o + |\dot{y}^2_t - \dot{y}^1_{t_i}|_o].$$

Similarly, as in the proof of the inequality (9) and (10), using the definitions of the norm $\|\cdot\|_o$ and $|\cdot|_o$ and the absolutely continuous function $y^2(\cdot)$ and making use of the inequality (20) we obtain

$$\|y^2_t - y^1_{t_i}\|_o \leq \varepsilon + \frac{2MT}{m} + \eta_1 T \text{ and } |y^2_t - y^1_{t_i}|_o \leq 2\varepsilon Mr.$$

Hence

$$H(F^2(t,y^2_t,\dot{y}^2_t),F^2(t,y^1_{t_i},\dot{y}^1_{t_i})) \leq$$

$$\leq k(\varepsilon + \frac{2MT}{m} + \eta_1 T + 2\varepsilon Mr).$$

By virtue of (16), we have

$$d(\dot{y}^2(t),\varepsilon F^2(t,y^2_t,\dot{y}^2_t)) \leq d(\dot{y}^2(t),\varepsilon F^2(t,y^1_{t_i},\dot{y}^1_{t_i})) +$$

$$+ H(\varepsilon F^2(t,y^1_{t_i},\dot{y}^1_{t_i}),\varepsilon F^2(t,y^2_t,\dot{y}^2_t)) \leq$$

$$\leq \varepsilon k(\varepsilon + \frac{2MT}{m} + \eta_1 T + 2\varepsilon Mr).$$

Now, on the ground of Filippov's type theorem (see Theorem 1) there exists the solution $x^2(\cdot)$ of (2) that for $t \in [0,T/\varepsilon]$

$$|y^2(t) - x^2(t)| \leq \varepsilon k(\varepsilon + \frac{2MT}{m} + \eta_1 T + 2\varepsilon Mr)\int_0^t e^{2\varepsilon k(t-s)}ds$$

$$\leq (\frac{\varepsilon}{2} + \frac{MT}{m} + \frac{\eta_1 T}{2} + \varepsilon Mr)(e^{2kT} - 1).$$

Hence, and by inequality (15) and (20), it follows

$$|x^1(t) - x^2(t)| \leq |x^1(t) - y^1(t)| + |y^1(t) - y^2(t)| +$$

$$+ |y^2(t) - x^2(t)| \leq \frac{2MT}{m} + (\varepsilon + 4\varepsilon Mr)(e^{kT} - 1) + \frac{2MT}{m} +$$

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$$+ \eta_1 T + \left(\frac{\varepsilon}{2} + \frac{MT}{m} + \frac{\eta_1 T}{2} + \varepsilon Mr\right)(e^{2kT} - 1) \leq$$

$$\leq \frac{MT}{m}(3 + e^{2kT}) + \frac{\eta_1 T}{2}(1 + e^{2kT}) + \frac{\varepsilon e^{2kT}}{2}(10Mr + 3).$$

Therefore, choosing

$$m > \frac{3MT(3 + e^{2kT})}{\eta}, \qquad \eta_1 = \frac{2\eta}{3T(1 + e^{2kT})}$$

and

$$\varepsilon < \frac{2\eta}{3e^{2kT}(10Mr + 3)},$$

we get the inequality

$$|x^1(t) - x^2(t)| \leq \eta \quad \text{for } t \in [0, T/\varepsilon].$$

Adopting now the procedure presented above we get condition (ii). In this way the proof is completed for $t \in [-r, T/\varepsilon]$.

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REZIME

TEOREMA O DELIMIČNOM USREDNJENJU ZA FUNKCIONALNO-DIFERENCIJALNE JEDNAČINE NEUTRALNOG TIPA

U ovom radu dokazana je jedna generalizacija rezultata Plotnikova iz [5] za funkcionalno-diferencijalne jednačine oblika $\dot{x}(t) \in F(t, x_t, \dot{x}_t)$ gde je F višeznačno preslikavanje sa vrednostima koje su neprazni kompaktni konveksni podskupovi n-dimenzionalnog prostora R^n.

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SOME RESULTS ON M- AND H- MATRICES

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ABSTRACT

In this paper some sufficient conditions for a matrix to be an M- or an H- matrix are given. The results include the ones from [1].

INTRODUCTION

We shall begin with some notations:

$N = \{1,2,\ldots,n\}$, $N(i) = N\setminus\{i\}$, $i \in N$.

For any matrix $A = [a_{ij}] \in C^{n,n}$ (= set of all the complex nxn matrices) and $i \in N$, $\alpha \in [0,1]$, we define

$$P_i(A) = \sum_{j\in N(i)} |a_{ij}|, \quad Q_i(A) = \sum_{j\in N(i)} |a_{ji}|, \quad R_i(A) = \sum_{j\in N} |a_{ij}|,$$

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$$P_{i,\alpha}(A) = \alpha P_i(A)+(1-\alpha)Q_i(A), \quad Q_i^*(A) = \max_{j\in N(i)} |a_{ji}|,$$

$$Q_i^{(r)}(A) = \max_{t_r\in e_r} \sum_{j\in t_r} |a_{ji}|,$$

where $r\in N$ and e_r is the set of all the choices $t_r=\{i_1,\dots,i_r\}$ of different indices from N.

Let $e_i: R_+^{2p(i)} \to R_+$, $p(i)\in N$, $i=1,2$, be two functions satisfying the following condition

(1) $x \geq y \Rightarrow e_i(x) \geq e_i(y)$, for any $x,y\in R_+^{2p(i)}$, $i=1,2$.

For any $A\in C^{n,n}$, $s\in N$ and $J = \{j_1,\dots,j_k\} \subset N$ we define

$$e_i(A,J,s) = e_i(R_{j_1}(A),\dots,R_{j_k}(A),Q_{j_1}^{(s)},\dots,Q_{j_k}^{(s)}(A)), \quad i=1,2.$$

Let s and J be fixed and $t_r\in e_r$. Then, we define

$$E(A,t_r) = e_1(A,t_r,s) + e_2(A,J,s).$$

Let $e_1^{(i)}$, $e_2^{(i)}$ and $E^{(k)}$ be functions of the above form and let $K(m,r) = \{(e_1^{(i)},e_2^{(i)}): i=1,\dots,m\}$. From now on we shall suppose that A=D-B, where D is diagonal part of A.

Definition 1. A matrix A is called K(m,r)-diagonally dominant iff for each $t_r\in e_r$ there exists an index $k\in\{1,\dots,m\}$, such that

$$E^{(k)}(D,t_r) > E^{(k)}(B,t_r).$$

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Definition 2. A set K(m,r) is called K-regular iff any K(m,r)-diagonally dominant matrix is regular.

Definition 3. A matrix A is called K(r(1),...,r(m))-diagonally dominant iff for each $j \in \{1,2,...,m\}$ and for each $t_{r(j)} \in \theta_{r(j)}$ it is fulfiled that

$$E^{(j)}(D,t_{r(j)}) > E^{(j)}(B,t_{r(j)}).$$

Definition 4. A set K(m,r) is called K_1-regular iff any K(r(1),...,r(m))-diagonally dominant matrix is regular.

Definition 5. A real square matrix whose off-diagonal elements are all non-positive is called an L-matrix.

Definition 6. A regular L-matrix A, for which $A^{-1} \geq 0$ is called an M-matrix.

For $C = [c_{ij}]$ and $F = \mathrm{diag}(f_1,...,f_n)$, we shall write $C \geq 0$ iff $c_{ij} \geq 0$ for each $i,j \in N$ and $F > 0$ iff $f_i > 0$ for each $i \in N$.

SUFFICIENT CONDITIONS FOR L-MATRICES TO BE M-MATRICES

Theorem 1. Let K(m,r) be a K-regular set. Let A be an L-matrix, whose diagonal elements are all positive. If A is a K(m,r)-diagonally dominant, then it is an M-matrix.

P r o o f : $A = D-B$, $D > 0$, $B \geq 0$ and $D^{-1}A = I - D^{-1}B$ where I

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is identity matrix. Let us prove that $\rho(D^{-1}B) < 1$. We shall suppose that there exists an eigenvalue λ of the matrix $D^{-1}B$, such that $|\lambda| \geq 1$. Then $\lambda D - B$ is a $K(m,r)$-diagonally dominant matrix, because for each $t_r \in \theta_r$ there exists an index $k \in (1,\ldots,n)$ such that

$$E^{(k)}(\lambda D, t_r) = E^{(k)}(|\lambda| D, t_r) \geq E^{(k)}(D, t_r) > E^{(k)}(B, t_r).$$

$K(m,r)$ is a K-regular set, so that the matrix $\lambda D - B$ is a regular matrix. But, then the matrix $D^{-1}(\lambda D - B) = \lambda I - D^{-1}B$ is a regular matrix, which contradicts the assumption that λ is an eigenvalue of $D^{-1}B$.

Hence, $\rho(D^{-1}B) < 1$, the matrix $I - D^{-1}B$ is regular and

$$(I - D^{-1}B)^{-1} = \sum_{i=0}^{\infty} (D^{-1}B)^i \geq 0.$$

Then A^{-1} exists and $A^{-1} = (I - D^{-1}B)^{-1} D \geq 0$.

Analogously we can prove the following theorem.

Theorem 2. Let A be an L-matrix, whose diagonal elements are all positive. Let $K(m,r)$ be a K_1-regular set. If A is a $K(r(1),\ldots,r(m))$- diagonally dominant matrix, then it is an M-matrix.

Theorem 3. Let A be an L-matrix, whose diagonal elements are all positive, such that at least one of the following conditions is satisfied:

(i) $a_{ii} > P_i(A)$, $i \in N$ (A is strictly diagonally dominant),

(ii) $a_{ii} > P_{i,\alpha}(A)$, $i \in N$, for some $\alpha \in [0,1]$,

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(iii) $a_{ii} > P_i^{\alpha}(A)Q_i^{1-\alpha}(A)$, $i \in N$, for some $\alpha \in [0,1]$,

(iv) $a_{ii}a_{jj} > P_i(A)P_j(A)$, $i \in N$, $j \in N(i)$,

(v) $a_{ii}a_{jj} > P_i^{\alpha}(A)Q_i^{1-\alpha}P_j^{\alpha}(A)Q_j^{1-\alpha}(A)$, $i \in N$, $j \in N(i)$,

for some $\alpha \in [0,1]$,

(vi) For each $i \in N$ it holds that

$a_{ii} > P_i(A)$ or

$a_{ii} + \sum_{j \in J} a_{jj} > Q_i(A) + \sum_{j \in J} Q_j(A)$, where $J := \{i \in N : a_{ii} \leq Q_i(A)\}$,

(vii) $a_{ii} > \min(P_i(A), Q_i^{*}(A))$, $i \in N$ and

$a_{ii} + a_{jj} > P_i(A) + P_j(A)$, $i \in N$, $j \in N(i)$,

(viii) $a_{ii} > Q_i^{(p)}(B)$, $i \in N$ and

$\sum_{j \in t_p} a_{ii} > \sum_{j \in t_p} P_i(A)$, $t_p \in e_p$, for some $p \in N$,

(ix) There exists $i \in N$ such that

$a_{ii}(a_{jj} - P_j(A) + |a_{ji}|) > P_i(A)|a_{ji}|$, $j \in N(i)$.

Then, A is an M-matrix.

P r o o f : (i) $a_{ii} = R_i(D) > R_i(B) = P_i(A)$, $i \in N$. Let r=1, m=1 and $e_1(x_1, x_2) = x_1$, $e_2 = \emptyset$. Then the matrix A is K(1,1)-diagonally dominant and $K(1,1) = \{(e_1, e_2)\}$ is a regular set. So, from Theorem 1, it follows that A is an M- matrix.

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Statements (ii)-(viii) can be proved similarly, by choosing

case	m	r	s	$e_1^{(i)}, i=1,\ldots,m$	$e_2^{(i)}, i=1,\ldots,m$
(i)	1	1	n	x_1	$\emptyset$
(ii)	1	1	n	$\alpha x_1+(1-\alpha)x_2$	$\emptyset$
(iii)	1	1	n	$x_1^{\alpha}x_2^{1-\alpha}$	$\emptyset$
(iv)	1	2	n	x_1x_2	$\emptyset$
(v)	1	2	n	$(x_1x_2)^{\alpha}(x_3x_4)^{1-\alpha}$	$\emptyset$
(vi)	2	1	n	x_1	$\emptyset$
				x_2	$\sum_{j\in J} x_{j+k}$, k=card J
(vii)	2	1	1	$\min(x_1,x_2)$	$\emptyset$
		2	n	x_1+x_2	$\emptyset$
(viii)	2	1	p	x_2	$\emptyset$
		p	p	$\sum_{j\in t_p} x_j$	$\emptyset$

(ix) can be proved in a similar way as in the proof of Theorem 1.

Note that strictly diagonally dominant matrices (SDD) satisfy all of the conditions (i)-(ix).

Statements (iii) and (iv) from Theorem 3 have been proved in [1].

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H-MATRIX CHARACTERIZATIONS

For any matrix $A=[a_{ij}]\in C^{n,n}$, we define $M(A)=[m_{ij}]\in R^{n,n}$ as follows

$$m_{ii} = |a_{ii}|, \; i\in N, \quad m_{ij} = -|a_{ij}|, \; i\in N, \; j\in N(i).$$

Definition 7. A matrix A is called an H-matrix iff M(A) is an M- matrix.

Definition 8. A matrix A is called generalized diagonally dominant (GDD) iff there exists a regular diagonal matrix W, so that AW is SDD.

Theorem 4. Let A be a matrix whose elements satisfy at least one of the conditions (i)-(ix) from Theorem 3, where all the diagonal elements of A are replaced by their modules. Then A is an H-matrix.

P r o o f : The matrix M(A) satisfies at least one of the conditions from Theorem 3 and it is an M-matrix.

Remark: Any irreducible diagonally dominant matrix is an H-matrix, too (see [2]).

Theorem 5. A matrix A is GDD if and only if it is an H-matrix.

P r o o f : Let A be GDD. Then, there exists a regular diagonal matrix W such that AW is SDD. Then, AW is an H- matrix, i.e. $M(AW)=M(A)M(W)$ is an M-matrix. Since M(W) is regular and $M(W) > 0$, it follows that

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$$(M(A))^{-1} = M(W)(M(AW))^{-1} \geq 0.$$

Conversely, if A is H-matrix, i.e. if M(A) is an M-matrix, then there exists a vector $z \in R^{n,n}$, $z > 0$, such that $M(A)z > 0$. It means that

$$|a_{ii}|z_i > \sum_{j \in N(i)} |a_{ij}|z_j \quad \text{for each } i \in N,$$

and we can choose the matrix $W = \text{diag}(z_1, \ldots, z_n)$.

Similarly we can prove the theorems analogous to Theorems 1 and 2 on the characterizatons of H-matrices.

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REZIME

NEKI REZULTATI O M- I H- MATRICAMA

Matrica koja ima nepozitivne vandijagonalne elemente naziva se L-matrica, a regularna L-matrica, čija inverzna matrica ima nenegativne elemente, naziva se M-matrica. Ako se od proizvoljne matrice A napravi L-matrica M(A), čiji su elementi po modulu isti sa elementima matrice A, tada se matrica A naziva H-matrica ako i samo ako je M(A) M-matrica. U radu su date neke nove karakterizacije M- i H- matrica, koje sadrže i neke od ranije poznatih rezultata, kao posebne slučajeve.

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THE SINGULARLY PERTURBED SPLINE COLLOCATION METHOD FOR BOUNDARY VALUE PROBLEMS WITH MIXED BOUNDARY CONDITIONS

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ABSTRACT

The spline collocations method given in [7] for solving boundary value problems without a singular perturbation is adapted for problems with singular perturbation. The exponential features of the exact solution are transferred to spline coefficients by „artifical viscousity". In this way a uniformly convergent method for solving problem: $\varepsilon y'' + p(x)y' + q(x)y = f(x), y'(0) - \alpha y(0) = \alpha_0, y(1) = \alpha_0, \alpha \geq 0, p(x) > 0, q(x) \equiv 0$ is achieved. The numerical results indicate a uniform convergence when $q(x) \neq 0$.

The following problem is considered:

$$(1)\quad \begin{cases} Ly = \varepsilon y'' + p(x)y' + q(x)y = f(x), \ 0 < x < 1, \\ y'(0) - \alpha y(0) = \alpha_0, \ y(1) = \alpha_1, \text{ where} \\ p(x) \geq p > 0, \ q(x) \leq 0; \ \alpha_1, \alpha_2, \alpha, \ \varepsilon \in R; \ \alpha \geq 0; \\ \varepsilon \in (0,1]. \end{cases}$$

AMS Mathematics Subject Classification (1980): 65L10, 65D07.
Key words and phrases: Spline collocation method, singular perturbation, artifical viscousity.

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The approximate solution to problem (1) is sought in the form of a cubic spline

$$S(x) = \sum_{i=-1}^{n+1} \beta_i B_i(x)$$

on the mesh $x_i = ih$, $i = -3(1)\overline{n+1}$, $n = 1/n$. $B_i(x)$ are basic splines determined in [7]. The coefficients β_i are determined from the conditions:

$$(3) \quad \begin{cases} L_h S(x_i) = \sigma(x_i)S''(x_i) + p(x_i)S'(x_i) + \\ \qquad + q(x_i)S(x_i) = f(x_i), \quad i = 0(1)n, \\ S'(0) - \alpha S(0) = \alpha_0 \\ S(1) = \alpha_1 \end{cases}$$

where $\sigma(x_i) = \{hp(x_i)/2\}cth\{hp(x_i)/2\varepsilon\}$.
By replacing (2) into (3) and using the characteristics of basic splines as in [7] we obtain the following system of linear equations:

$$(4) \quad \begin{cases} \beta_{i-1}K_i + \beta_i L_i + \beta_{i+1}M_i = f(x_i), \quad i = 0(1)n, \\ \beta_{-1}(h+3\alpha) + 4\beta_0 h + \beta_1(h-3\alpha) = 6\alpha_0 h \\ \beta_{n-1} + 4\beta_n + \beta_{n+1} = 6\alpha_1 \end{cases}$$

where

$$K_i = \frac{1}{h^2}\left(\sigma(x_i) - \frac{p(x_i)h}{2} + \frac{q(x_i)h^2}{6}\right),$$

$$M_i = \frac{1}{h^2}\left(\sigma(x_i) + \frac{p(x_i)h}{2} + \frac{q(x_i)h^2}{6}\right),$$

$$L_i = -M_i - K_i + q(x_i).$$

Excluding the unknown β_{-1} and β_{n+1} from the above equations for $i = 0$ and $i = n$, we obtain the system:

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$$(5)\quad \begin{cases} \beta_0 \bar{L}_0 + \beta_1 \bar{M}_0 = d_0 \\ \beta_{i-1} K_i + \beta_i L_i + \beta_{i+1} M_i = f(x_i), \ i = 1(1)\overline{n-1}. \\ \beta_{n-1} \bar{K}_n + \beta_n \bar{L}_n = d_n, \end{cases}$$

where

$$\bar{L}_0 = L_0 - 4hK_0/(h+3\alpha), \quad \bar{M}_0 = M_0 - (h-3\alpha)K_0/(h+3\alpha),$$

$$d_0 = f(x_0) - 6\alpha_0 h/(h+3\alpha), \quad \bar{K}_n = K_n - M_n,$$

$$\bar{L}_n = L_n - 4M_n, \quad d_n = f(x_n) - 6\alpha_1 M_n.$$

System (6) can be written in the form $A\beta = d$, where A is a matrix of the type $(n+1) \times (n+1)$ with elements a_{ij} $(i,j = 0(1)n)$, β and d are vectors with elements β_i, and d_i; $d_i = f(x_i)$, $i = 1(1)n$.
The matrix A is diagonally dominant and irreducible. System (4) has a unique solution which we can obtain by the methods given in [4].

In order to prove the uniform convergence in the case $q(x) \equiv 0$, we shall derive the corresponding difference scheme. On each interval $[x_j, x_{j+1}]$, the spline S(x) has the form:

$$(6)\quad S(x) = v_j(x) = v_j^{(0)} + (x - x_j)v_j^{(1)} + \frac{(x - x_j)^2 v_j^{(2)}}{2!} + \frac{(x - x_j)^3 v_j^{(3)}}{3!}, \qquad x \in [x_j, x_{j+1}].$$

Constants $v_j^{(k)}$, $k = 0,1,2,3$, satisfy the following system:

$$(7)\quad \sigma_j v_j^{(2)} + p_j v_j^{(1)} = f_j, \quad j = 0(1)\overline{n-1}$$

$$\sigma_n (v_{n-1}^{(2)} + h v_n^{(3)}) + p_n (v_{n-1}^{(1)} + h v_{n-1}^{(2)} + h^2 v_{n-1}^{(3)}/2) = f_n.$$

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$$v_j^{(0)} = v_{j-1}^{(0)} + v_{j-1}^{(1)}h + h^2 v_{j-1}^{(2)}/2 + h^3 v_{j-1}^{(3)}/6, \quad j = 1(1)\overline{n-1}.$$

$$v_j^{(1)} = v_{j-1}^{(1)} + h v_{j-1}^{(2)} + h^2 v_{j-1}^{(3)}/2, \quad j = 1(1)\overline{n-1}$$

(8) $$v_j^{(2)} = v_{j-1}^{(2)} + h v_{j-1}^{(3)}, \quad j = 1(1)\overline{n-1}$$

(9) $$v_0^{(1)} - \alpha v_0^{(0)} = \alpha_0, \quad v_n^{(0)} = \alpha_1, \text{ where}$$

$$\sigma_j = \sigma(x_j), \quad p_j = p(x_j), \quad f_j = f(x_j).$$

By the elimination of $v_j^{(1)}$, $v_j^{(2)}$ and $v_j^{(3)}$, we obtain the scheme (see [3]):

(10) $$R_h v_j = Q_h f_j, \quad j = 0(1)\overline{n-1}, \text{ where}$$

$$v_j = v_j^{(0)}$$

$$R_h v_j = r_j^- v_{j-1} + r_j^c v_j + r_j^+ v_{j+1}, \quad j = 1(1)\overline{n-2}$$

$$Q_h f_j = q_j^- f_{j-1} + q_j^c f_j + q_j^+ f_{i+1}, \quad j = 1(1)\overline{n-2}$$

$$R_h v_0 = -(1/\gamma_1 + \alpha)v_0 + (1/\gamma_1)v_1$$

$$Q_h f_0 = \alpha_0 + [h^2 f_0(\omega_1 + 1)/(6\sigma_0(\omega_1 + 1)) + h^2 f_1 \omega_1/(6\sigma_1(\omega_1 + 1))]/\gamma_1$$

$$R_h v_{n-1} = r_{n-1}^- v_{n-2} + r_{n-1}^c v_{n-1}$$

$$Q_h f_{n-1} = q_{n-1}^- f_{n-2} + q_{n-1}^c f_{n-1} + q_{n-1}^+ f_n - r_{n-1}^+ \alpha_1$$

$$r_j^- = 3(\omega_{j-1} - 1)/(hA_{j-1}), \quad r_j^+ = 3(\omega_{j+1}+1)/(hA_j)$$

$$r_j^c = -r_j^+ - r_j^-$$

$$q_j^- = 1/(p_{j-1}A_{j-1}), \quad q_j^+ = 1/(p_{j+1}A_j)$$

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$$q_j^c = [(2\omega_{j-1} - 1)/A_{j-1} + (2\omega_{j+1} + 1)/A_j]/(\omega_j p_j)$$

$$\sigma_j = \varepsilon\rho_j\omega_j, \quad \omega_j = c\,th(p(x_j)/2\varepsilon)),$$

$$A_j = 3\omega_j\omega_{j+1} + 2\omega_j - 2\omega_{j+1} - 1.$$

$$\gamma_{j+1} = hA_j/(3\omega_j(\omega_{j+1} + 1))$$

At the same time, we have that

$$\bar{R}_h v_j^{(1)} = \bar{Q}_h f_j, \quad j = 1(1)n. \tag{11}$$

$$\bar{R}_h = a_j v_j^{(1)} - b_j v_{j-1}^{(1)}, \quad \bar{Q}_h f_j = \frac{h}{2}\left(\frac{f_{j-1}}{\sigma_{j-1}} + \frac{f_j}{\sigma_j}\right)$$

$$a_j = 1 + \frac{hp_j}{2\sigma_j}, \qquad b_j = 1 - \frac{hp_{j-1}}{2\sigma_{j-1}}.$$

Before proving the uniform convergence, we shall show some important properties of operators R_h and $\bar{R}_h$. The local truncation errors of these operators are defined in the usual way, i.e.,

$$\tau_j(y) = R_h(y(x_j) - v_j) = R_h y(x_j) - Q_h(Ly(x_j))$$

$$\bar{\tau}_j(y') = \bar{R}_h(y'(x_j) - v_j^{(1)}) = R_h y'(x_j) - \bar{Q}_h(Ly(x_j)).$$

If $y(x) = exp(-px/\varepsilon)$ and $p(x) = p = const.$, then $\tau_j = 0$, $j = 1(1)\overline{n-1}$, $\bar{\tau}_j = 0$, $j = 1(1)n$. This is the consequence of the fact that in this case

$$r_j^-/r_j^+ = exp(-ph/\varepsilon), \quad a_j/b_j = exp(-ph/\varepsilon).$$

The scheme of this type for the Dirichlet's boundary conditions has been derived in [5]. It has the first order of the uniform convergence. The same has been proved in [6] for scheme (10). We shall outline that proof and show that the

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same holds for the first derivative (Theorem 1). The convergence between grid points we consider in Theorem 2. The following lemmas are used in the comparison function proof to bound the truncation error (see [1]).

Lemma 1. ([6]) *Let* $\{v_j\}$ *be a set of values at the grid points* x_j *satisfying* $R_h v_j \geq 0$, $j = 0(1)\overline{n-1}$. *Then* $v_j \leq 0$, $j = 0(1)\overline{n-1}$.

Lemma 2. ([6]) *Let* $p(x), q(x) \in C^3[a,b]$. *Then the solution of (1) can be written in the form*

$$y(x) = u(x) + w(x),$$

where

$$u(x) = \delta_1 \varepsilon \exp(-p(0)x/\varepsilon),$$

$$|w^{(i)}(x)| \leq M(1 + \varepsilon^{-i+2}\exp(-2\delta_0 x/\varepsilon)), (i = 1,2,3,4),$$

δ_0, δ_1 *and* M *are constants independent of* ε.

Lemma 3. ([6]) *There are constants* c_1 *and* c_2 *independent of* h *and* ε, *such that for* $h \leq c_1$, $0 < \delta \leq c_2$, $j = 1(1)\overline{n-1}$,

a) $R_h \varphi_j \geq M\frac{h^2}{\varepsilon^2}$ *for* $h \leq \varepsilon$,

b) $R_h \varphi_j \geq M$ *for* $\varepsilon \leq h$,

c) $R_h \psi_j \geq M\mu^j(\delta)/h$ *for* $\varepsilon \leq h$,

d) $R_h \varphi_j \geq M\frac{h^2}{\varepsilon^2}\frac{1}{\varepsilon}\mu^j(\delta)$ *for* $h \leq \varepsilon$,

e) $R_h \varphi_0 \geq M$

f) $R_h \psi_0 \geq M\varepsilon^{-1}\exp(-t_0\delta)$ *for* $h \leq \varepsilon$,

g) $R_h \psi_0 \geq Mh^{-1}\exp(-t_0\delta)$ *for* $\varepsilon \leq h$,

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$$t_j = x_j/\varepsilon,\ \varphi_j = -2 + x_j,\ \psi_j = -\exp(-\delta t_j),$$

$$\mu(\delta) = \exp(-\delta h/\varepsilon).$$

Corollary 1. *If* $k_1(h,\varepsilon) \geq 0$ *and* $k_2(h,\varepsilon) \geq 0$ *are such functions that*

$$R_h(k_1\varphi_j + k_2\psi_j) \geq R_h(\pm z_j) = \pm\tau_j(y),\quad j = \overline{0(1)n-1}$$

then

$$|z_j| \leq k_1|\varphi_j| + k_2|\psi_j|,\quad z_j^{(k)} = y^{(k)}(x_j) - v_j^{(k)},\ k=0,1,2,3.$$

Throughout the paper, M (or δ) denotes the possible different constants independent of ε and h.

Theorem 1. *Let* $\{v_j\}$ *and* $\{v_j^{(1)}\}$ *be the approximation to* $y(x_j)$ *and* $y'(x_j)$, $j = \overline{0(1)n}$, *obtained by using (10) and (11), respectively. Let* $q(x) \equiv 0$, $p(x) \in C^3[0,1]$, $p(x) \geq p > 0$, $\alpha \geq 0$.

Then

$$|y(x_j) - v_j| + |y^{(1)}(x_j) - v_j^{(1)}| \leq Mh^2/(\varepsilon + h). \tag{12}$$

Proof. In the main part of the proof we shall estimate the truncation error for u(x) and w(x) separately.

$$\tau_j(y) = \tau_j(u) + \tau_j(w),\quad z_j^{(k)} = z_j^{(k)}(u) + z_j^{(k)}(w),$$

$$\bar{\tau}_j(y^{(1)}) = \bar{\tau}_j(u^{(1)}) + \bar{\tau}_j(w^{(1)})$$

In [6] it has been shown that

$$|\tau_j(y)| \leq \begin{cases} M\dfrac{h^4}{\varepsilon^2(h+\varepsilon)} + \dfrac{h^2}{\varepsilon}\exp(-\delta t_j),\ h \leq \varepsilon,\ j = \overline{1(1)n-1} \\ M(h + \varepsilon\exp(-\delta t_{j-1}),\ \varepsilon \leq h \end{cases}$$

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$$|\tau_o(w)| \le \begin{cases} M(\frac{h^2}{\varepsilon+h} + \frac{h^3}{\varepsilon^2} exp(-\delta to)), & h \le \varepsilon. \\ M(\frac{h^2}{\varepsilon+1} + h\ exp(-\delta to)), & \varepsilon \le h, \end{cases}$$

$$|\tau_o(u)| \le \begin{cases} Mh^2 exp(-\delta to)/\varepsilon^2, & h \le \varepsilon, \\ M\ exp(-\delta to), & \varepsilon \le h, \end{cases}$$

and by Corollary 1 and Lemma 3 we have that the contribution to the error from $\tau_j(y)$ satisfies (12). As in [3], we have

$$\bar{\tau}_j(y') = \bar{\tau}_j(y', p_j, p_{j-1}) = R_2(x_{j-1}, x_j, y') -$$

$$- hR_1(x_{j-1}, x_j, y'')/2 + h(\eta_{j-1}/\sigma_{j-1} + \eta_j/\sigma_j)/2,$$

where

$$R_i(x_{j-1}, x_j, g) = g^{(i+1)}(\zeta) \frac{(x_j - x_{j-1})^{i+1}}{(i+1)!} =$$

$$= \frac{1}{i!} \int_{x_{j-1}}^{x_j} (x_j - s)^i g^{(i+1)}(s) ds$$

$$x_{j-1} \le \zeta \le x_j, \quad \eta_j = y''(x_j)(\sigma_j - \varepsilon).$$

Since $(\sigma_j - \varepsilon) \le Mh^2/(\varepsilon+h)$, we have

$$|\bar{\tau}_j(\omega', p_j, p_{j-1})| \le Mh^2/(\varepsilon+h) + h^3 exp(-\delta t_j)/\varepsilon^2, \quad h \le \varepsilon. \tag{13}$$

$$\bar{\tau}_j(u', p_j, p_{j-1}) = \bar{\tau}_j(u', p_j, p_{j-1}) - \bar{\tau}_j(u', p(0), p(0))$$

because

$$\bar{\tau}_j(u', p(0), p(0)) = 0$$

$$|\bar{\tau}_j(u', p_j, p_{j-1}) - \bar{\tau}_j(u', p(0), p(0))| \le$$

$$\le h|(\eta_{j-1}/\sigma_{j-1} - \eta_o/\sigma_o + \eta_j/\sigma_j - \eta_o/\sigma_o)|/2.$$

After the Taylor development we obtain

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(14) $$|\bar{\tau}_j(u', p_j, p_{j-1})| \le M \frac{h}{\varepsilon}\left\{\frac{x_{i-1}h^2u(x_{j-1})}{\sigma_0\sigma_{j-1}} + \right.$$

$$\left. + \frac{x_i h^2 u(x_j)}{\sigma_0\sigma_j}\right\} \le M \frac{h^3}{\varepsilon^2}\exp(-\delta t_j),\ h \le \varepsilon,\ j = 1(1)n.$$

For $j = 1$, the first term is equal to zero.
Since

(15) $$z_0^{(1)} - \alpha z_0^{(0)} = 0,$$ we have that $|z_0^{(1)}| \le Mh^2/(\varepsilon+h)$

and from (13), (14) and

(16) $$a_j z_j^{(1)} = b_j z_{j-1}^{(1)} + \bar{\tau}_j(y', p_j, p_{j-1}),\ j = 1(1)n$$

we have that (11) holds for $h \le \varepsilon$.

Let $\varepsilon \le h$. Then we consider $z_j^{(1)}(u)$ and $z_j^{(1)}(w)$.
From

(17) $$z_j^{(2)} = (\eta_j - p_j z_j^{(1)})/\sigma_j\ ,$$ and

$$z_j^{(1)} = z_{j-1}^{(1)} + h z_{j-1}^{(2)} + R_1(x_{j-1}, x_j, z')$$

we have

(18) $$z_j^{(1)} = z_{j-1}^{(1)}(1 - hp_{j-1}/\sigma_{j-1}) + h\eta_{j-1}/\sigma_{j-1} +$$

$$+ R_1(x_{j-1}, x_j, z').$$

Using the integral form of the remainder term, we have

$$|R_1(x_{j-1}, x_j, w')| \le Mh(h + \exp(-\delta t_{j-1})),\ j = 1(1)n.$$

From (17), (15) and $|v_j^{(3)}(w)| \le Mh^{-1}$, we obtain

(19) $$|z_j^{(1)}(w)| \le Mh^2/(\varepsilon+h).$$

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For $z_j^{(1)}(u)$ we take form (16). Since

$$\sigma_0 - \sigma_j = \omega_j(hp(0) - hp(x_i))/2 + hp(0)(\omega_0 - \omega_j)/2$$

$$|\sigma_0 - \sigma_j| \le Mhx_i,$$

we have

$$|\bar{\tau}_j(u', p_j, p_{j-1})| \le Mh\exp(-\delta t_{j-1}), \quad j = 1(1)n. \tag{20}$$

Then from (15), (16), (19) and (20) there follows (11).

Theorem 2. *Let the conditions of Theorem 1 be satisfied. Then*

$$|y(x) - S(x)| \le Mh^2/(\varepsilon+h), \tag{21}$$

M *is a constant independent of* h *and* ε.

Proof. The function $z(x) = y(x) - S(x)$ has the form

$$z(x) = z_j + (x - x_j)z_j^{(1)} + \frac{(x-x_j)^2}{2!} z_j^{(2)} + \frac{(x-x_j)^3}{3!} z_j^{(3)} + R_3(x_{j-1}, x_j, y), \tag{22}$$

on each subinterval $x \in [x_j, x_{j+1}]$.

Because of

$$z_j^{(3)} = (z_j^{(2)} - z_{j-1}^{(2)} - R_1(x_{j-1}, x_j, y''))/h$$

and

$$|y^{(i)}| \le M(1 + \varepsilon^{-i+1}\exp(-\delta t_j)),$$

from (17) we have

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$$|z_j^{(i+2)}| \leq M \frac{h^{-i}}{\varepsilon+h}, \quad i = 0,1$$

and $j = 0(1)n$ if $h \leq \varepsilon$, $j = 1(1)n$ if $\varepsilon \leq h$.

Thus from (22), we obtain (21) for $j = 1(1)n$. To complete the proof we must show that the same holds for $\varepsilon \leq h$ and $j = 0$.

Indeed,

$$|z(x)| \leq |z_0| + |(x - x_0)z'(\zeta)| \leq Mh, \quad \zeta \in [x_0, x_1]$$

because

$$|z_i^{(1)}| \leq Mh \Rightarrow |v_i^{(k)}| \leq Mh^{-k+1}, \quad k = 1,2,3$$

and

$$|v_0^{(1)}(x)| \leq M, \quad x \in [x_0, x_1] \quad \text{(see (6), (7), (8))},$$

$$|z'(\zeta)| = |y'(\zeta) - v_0'(\zeta)| \leq M$$

Numerical example [2]:

$$\varepsilon u'' + (1 + x^2)u' - (x - \tfrac{1}{2})^2 u =$$

$$= -4(3x^2 - 3x + 1)((x - \tfrac{1}{2})^2 + 2)$$

$$u(0) - u'(0) = 0, \quad u(1) = 0.$$

Our test for the order of uniform convergence and the notation are taken from [2]. Table 1 contains the results for the solution and Table 2 for the first derivative.
The computed order of uniform convergence is 1.20 and the classical one 2.02, (Table 1).
The computed estimate of the order of uniform convergence is 1.22 and the classical one 2.00, (Table 2).

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Table 1.

$\varepsilon\backslash k$	0	1	2	3	4	$\bar{p}_\varepsilon$
1/2	1.99	2.00	2.00	1.97	2.12	2.02
1/4	1.95	2.00	2.00	2.00	1.99	1.99
1/8	1.92	1.97	1.99	2.00	2.07	1.99
1/16	1.87	1.87	1.96	1.99	1.96	1.93
1/32	1.89	1.68	1.91	1.98	1.99	1.89
1/64	1.50	1.59	1.74	1.92	1.98	1.75
1/128	1.10	1.53	1.42	1.76	1.93	1.55
1/256	1.03	1.09	1.47	1.40	1.77	1.35
1/512	1.03	1.02	1.09	1.42	1.42	1.20

Table 2.

$\varepsilon\backslash k$	0	1	2	3	4	$\bar{p}_\varepsilon$
1/2	2.00	2.00	2.00	2.00	2.00	2.00
1/4	2.01	2.00	2.00	2.00	2.00	2.00
1/8	2.02	2.00	2.00	2.00	2.00	2.00
1/16	2.02	1.99	2.00	2.00	2.00	2.00
1/32	1.82	1.95	1.99	2.00	2.00	1.95
1/64	1.29	1.83	1.92	1.98	2.00	1.80
1/128	1.00	1.32	1.81	1.93	1.98	1.61
1/256	0.95	1.03	1.33	1.80	1.92	1.41
1/512	0.95	0.98	1.04	1.34	1.80	1.22

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REZIME

SINGULARNO PERTURBIRAN SPLINE KOLOKACIONI METOD ZA RUBNI PROBLEM SA MEŠOVITIM RUBNIM USLOVIMA

Spline kolokacioni metod dat u [7] za rešavanje rubnog problema bez singularne perturbacije je prilagodjen za problem sa singularnom perturbacijom. Eksponencijalne karakteristike tačnog rešenja su prenete na splajn koeficijente pomoću "veštačke viskoznosti". Na taj način dobijen je uniformno konvergentan metod za rešavanje problema:

$$\varepsilon y'' + p(x)y' + q(x)y = f(x), y'(0) - \alpha y(0) = \alpha_0, \; y(1) = \alpha_1,$$

$$\geq 0, \quad p(x) > 0, \; q(x) \equiv 0.$$

Numerički rezultati ukazuju na uniformnu konvergenciju kada je $q(x) \neq 0$.

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SOME RELATIONS FOR CURVATURE TENSORS IN A FINSLER SPACE

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ABSTRACT

This paper is a continuation of previous papers [5] and [6] by the same author. In [6] the induced connection coefficients which appear in (1.13), (1.14) and (1.15) are determined under conditions when $\bar{D}\ell^d$ and $\bar{D}\ell^k$ are defined by (1.18) and (1.19). In [6] it is proved that the mentioned formulae are consistent with each other only when relation (1.21) is satisfied. This condition is satisfied in the several cases. In this paper we shall examine the special case when $B^\alpha_a = B^\alpha_a(x)$ and $N^\alpha_k = N^\alpha_k(x)$ i.e. when B^α_a and N^α_k are not functions of $\dot{x}$. Since we suppose (1.1) i.e. that $g_{\alpha\beta}(x,\dot{x})\, B^\alpha_a(x)\, N^\alpha_k(x) = 0$ so our examination is restricted only to those Finsler spaces in which the metric tensor has such a special form that relation (1.1) is valid. Let us denote such Finsler spaces by $\bar{F}_n$. The curvature tensors in $\bar{F}_n$ are defined by (2.5), (2.6) and (2.7). In this paper the relations between alternated differentials of a vector field and the curvature tensors are given. The curvature tensors and their alternated differentials are decomposed in the direction of vectors B^α_a and N^α_k.

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PRELIMINARIES

In the Finsler space $\overline{F}_n$ the metric function is $L(x,\dot{x})$. Let us define m fields of vectors $B_a^\alpha(x)$ and n-m fields $N_k^\alpha(x)$ ($\alpha,\beta,\gamma,\delta,\varepsilon,\kappa,\ldots = 1,2,\ldots,n$; $a,b,c,d,e,f,i,j = 1,2,\ldots,m$; $k,l,m,n,p,q = m+1,\ldots,n$) in such a way that these vector fields are linearly independent at each x and satisfy the relations

$$(1.1)\qquad g_{\alpha\beta}B_a^\alpha N_k^\beta = 0 \quad \text{for each} \quad a = 1,2,\ldots,m, \quad k = m+1,\ldots,n.$$

Let us define

$$(1.2)\qquad g_{ab} = g_{\alpha\beta}B_a^\alpha B_b^\beta$$

$$(1.3)\qquad g_{kl} = g_{\alpha\beta}N_k^\alpha N_l^\beta$$

$$(1.4)\qquad B_\beta^b = g^{ab}\, g_{\alpha\beta}B_a^\alpha$$

$$(1.5)\qquad N_\beta^k = g^{km}\, g_{\alpha\beta}N_m^\alpha$$

$g_{\alpha\beta}$, B_α^a and N_α^k have zero degree of homogeneity in $\dot{x}$, (g^{ab}) and (g^{km}) are inverse matrices of (g_{ab}) and (g_{km}), respectively. From (1.3) and (1.5) we have

$$(1.6)\qquad N_\alpha^k N_p^\alpha = g^{kl}\, g_{\alpha\beta}N_l^\beta N_p^\alpha = g^{kl}g_{lp} = \delta_p^k.$$

Usually, we have that:

$$(1.7)\qquad \delta_\beta^\alpha = B_a^\alpha B_\beta^a + N_k^\alpha N_\beta^k.$$

The vectors dx and $\dot{x}$ are decomposed in the direction of vectors B_a^α and N_k^α in the following way

$$(1.8)\qquad dx^\alpha = B_a^\alpha\, du^a + N_k^\alpha\, dv^k$$

$$(1.9)\qquad \dot{x}^\alpha = B_a^\alpha\, \dot{u}^a + N_k^\alpha\, \dot{v}^k$$

We shall suppose that

$$0 = F^\alpha(x^1,\ldots,x^n,\ u^1,\ldots,u^m,\ v^{m+1},\ldots,v^n) \qquad \alpha = 1,2,\ldots,n,$$

any of the solutions of system of differential equation (1.8), to-

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gether with (1.9) define x and $\dot{x}$ as the function of u, $\dot{u}$, v, $\dot{v}$ in the form

$$x^\alpha = x^\alpha\ (u^1, \ldots,\ u^m,\ v^{m+1}, \ldots, v^n)$$

$$\dot{x}^\alpha = \dot{x}^\alpha\ (u^1, \ldots, u^m,\ v^{m+1}, \ldots, v^n, \dot{u}^1, \ldots \dot{u}^m, \dot{v}^{m+1}, \ldots, \dot{v}^n)$$

$$\alpha = 1, 2, \ldots, n$$

We shall suppose that the tensor and vector fields are homogeneous of degree zero in $\dot{x}$. For any vector field ξ, we have

$$\xi^\alpha = B^\alpha_a\ \xi^a + N^\alpha_k\ \xi^k. \tag{1.10}$$

Let us denote the absolute differential which corresponds to the motion from $(x, \dot{x})$ to $(x + dx,\ \dot{x} + d\dot{x})$ by D. Then, we have

$$D\xi^\alpha = (DB^\alpha_a)\xi^a + B^\alpha_a\ d\xi^a + (DN^\alpha_k)\xi^k + N^\alpha_k\ d\xi^k. \tag{1.11}$$

We shall use the notation

$$\ell^\alpha = L^{-1}(x,\dot{x})\ \dot{x}^\alpha = L^{-1}(B^\alpha_a \dot{u}^a + N^\alpha_k\ \dot{v}^k) = B^\alpha_a \ell^a + N^\alpha_k \ell^k \tag{1.12}$$

where $\ell^a = L^{-1}\ \dot{u}^a$ and $\ell^k = L^{-1}\dot{v}^k$. From [6] we have

$$DB^\alpha_a = \bar{W}^d_a(d) B^\alpha_d + \bar{W}^k_a(d) N^\alpha_k, \tag{1.13}$$

$$DN^\alpha_m = \bar{W}^d_m(d) B^\alpha_d + \bar{W}^k_m(d) N^\alpha_k, \tag{1.14}$$

where

$$\bar{W}^y_x(d) = \bar{\Gamma}^{*\ y}_{x\ \ b}\ du^b + \bar{\Gamma}^{*\ y}_{x\ \ n}\ dv^n + \bar{A}^{\ y}_{x\ \ b}\ \bar{D}\ell^b + \bar{A}^{\ y}_{x\ \ n}\ \bar{D}\ell^n \tag{1.15}$$

$$x = a \text{ or } x = m,\quad y = d \text{ or } y = k.$$

The induced differentials $\bar{D}\xi^a$, $\bar{D}\xi^k$ are defined by $\bar{D}\xi^a = B^a_\alpha\ D\xi^\alpha$ $\bar{D}\xi^k = N^k_\alpha\ D\xi^\alpha$ and

$$D\xi^\alpha = B^\alpha_d\ \bar{D}\xi^d + N^\alpha_k\ \bar{D}\xi^k. \tag{1.16}$$

For ℓ^α we have

$$D\ell^\alpha = B^\alpha_d\ \bar{D}\ell^d + N^\alpha_k\ \bar{D}\ell^k, \tag{1.17}$$

where $\bar{D}\ell^d$ and $\bar{D}\ell^k$ will be defined by

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$$(1.18)\qquad \bar{D}\ell^d = d\ell^d + \bar{\Gamma}^{*d}_{o\,c}\, du^c + \bar{\Gamma}^{*d}_{o\,\ell}\, dv^\ell$$

$$(1.19)\qquad \bar{D}\ell^k = d\ell^k + \bar{\Gamma}^{*k}_{o\,c}\, du^c + \bar{\Gamma}^{*k}_{o\,\ell}\, dv^\ell$$

$$(1.20)\qquad \bar{\Gamma}^{*x}_{o\,y} = \bar{\Gamma}^{*x}_{a\,y}\ell^a + \bar{\Gamma}^{*x}_{m\,y}\ell^m = L^{-1}\bar{\Gamma}^{*x}_{y}$$

$$x = d \text{ or } x = k, \qquad y = c \text{ or } y = \ell.$$

As it was proved in [6], relation (1.17) is consistent with (1.18) - (1.20) iff

$$(1.21)\qquad \left[(\dot{\partial}_\beta B^\alpha_a)\,\dot{u}^a + (\dot{\partial}_\beta N^\alpha_k)\,\dot{v}^k\right] D\ell^\beta = 0.$$

In $\bar{F}_n$ this condition is obviously satisfied.

In [6] the induced connection coefficients $\bar{\Gamma}$ and $\bar{A}$ are determined. From $\ell_\alpha\, D\ell^\alpha = 0$, using (1.1), (1.17), we obtain

$$g_{\alpha\beta}\,(B^\alpha_a\ell^a + N^\alpha_k\ell^k)(B^\beta_b\,\bar{D}\ell^b + N^\beta_\ell\,\bar{D}\ell^\ell) = g_{ab}\ell^a\,\bar{D}\ell^a + g_{k\ell}\ell^k\,\bar{D}\ell^k = 0$$

i.e.

$$(1.22)\qquad \ell_b\,\bar{D}\ell^b + \ell_\ell\,\bar{D}\ell^\ell = 0.$$

For $\bar{D}\xi^a$ and $\bar{D}\xi^k$, we have

$$(1.23)\qquad \bar{D}\xi^a = B^a_\alpha\, D\xi^\alpha = d\xi^a + \bar{w}^a_b(d)\,\xi^b + \bar{w}^a_k(d)\,\xi^k$$

$$(1.24)\qquad \bar{D}\xi^k = N^k_\alpha\, D\xi^\alpha = d\xi^k + \bar{w}^k_a(d)\,\xi^a + \bar{w}^k_m(d)\,\xi^m.$$

From (1.16) it is obvious that $\bar{D}\xi^a$ and $\bar{D}\xi^k$ are components of $D\xi^\alpha$ in the directions of B^α_a and N^α_k, respectively.

2. ALTERNATED DIFFERENTIALS EXPRESSED BY CURVATURE TENSORS

If Δ is another absolute differential, corresponding to the motion from $(x,\dot{x})$ to $(x+\delta x, \dot{x}+\delta\dot{x})$, then we have

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$$[\Delta D]\xi^{\alpha} = ([\Delta D]B^{\alpha}_{a})\xi^{a} + B^{\alpha}_{a}[\delta d]\xi^{a} + ([\Delta D]N^{\alpha}_{k})\xi^{k} + N^{\alpha}_{k}[\delta d]\xi^{k}, \tag{2.1}$$

where

$$[\Delta D]B^{\alpha}_{a} = \bar{\Omega}_{a}^{e}(d\delta)B^{\alpha}_{e} + \bar{\Omega}^{m}_{a}(d\delta)N^{\alpha}_{m} + \mathcal{D}B^{\alpha}_{a} \tag{2.2}$$

$$[\Delta D]N^{\alpha}_{k} = \bar{\Omega}^{e}_{k}(d\delta)B^{\alpha}_{e} + \bar{\Omega}^{m}_{k}(d\delta)N^{\alpha}_{m} + \mathcal{D}N^{\alpha}_{k} \tag{2.3}$$

$$\bar{\Omega}_{x}^{y}(d,\delta) = \frac{1}{2}\bar{R}_{x\ bc}^{\ y}[du^{b}\delta u^{c}] + \bar{R}_{x\ bk}^{\ y}[du^{b}\delta v^{k}] + \frac{1}{2}\bar{R}_{x\ k\ell}^{\ y}[dv^{k}\delta v] + \tag{2.4}$$

$$\bar{P}_{x\ bc}^{\ y}[du^{b}\bar{\Delta}\ell^{c}] + \bar{P}_{x\ bk}^{\ y}[du^{b}\bar{\Delta}\ell^{k}] + \bar{P}_{x\ kc}^{\ y}[dv^{k}\bar{\Delta}\ell^{c}] + \bar{P}_{x\ k\ell}^{\ y}[dv^{k}\bar{\Delta}\ell^{\ell}] +$$

$$\frac{1}{2}\bar{S}_{x\ bc}^{\ y}[\bar{D}\ell^{b}\bar{\Delta}\ell^{c}] + \bar{S}_{x\ bk}^{\ y}[\bar{D}\ell^{b}\bar{\Delta}\ell^{k}] + \frac{1}{2}\bar{S}_{x\ k\ell}^{\ y}[\bar{D}\ell^{k}\bar{\Delta}\ell^{\ell}]$$

where $x \in \{a, k\}$ $y \in \{e, m\}$,

$$\bar{R}_{x\ zw}^{\ y} = 2((\partial_{[w}\bar{\Gamma}^{*y}_{|x|\ z]} - \dot{\partial}_{d}\bar{\Gamma}^{*y}_{x[z}\bar{\Gamma}^{*d}_{w]} - \dot{\partial}_{\ell}\bar{\Gamma}^{*y}_{x[z}\bar{\Gamma}^{*\ell}_{w]} + \tag{2.5}$$

$$\bar{\Gamma}^{*d}_{x[z}\bar{\Gamma}^{*y}_{|d|w]} + \bar{\Gamma}^{*\ell}_{x[z}\bar{\Gamma}^{*y}_{|\ell|w]}) +$$

$$L^{-1}A_{x}^{\ y}{}_{d}(\partial_{[w}\bar{\Gamma}^{*d}_{z]} - \dot{\partial}_{f}\bar{\Gamma}^{*d}_{[z}\bar{\Gamma}^{*f}_{w]} - \dot{\partial}_{\ell}\bar{\Gamma}^{*d}_{[z}\bar{\Gamma}^{*\ell}_{w]}) +$$

$$L^{-1}A_{y}^{\ y}{}_{k}(\partial_{[w}\bar{\Gamma}^{*k}_{z]} - \dot{\partial}_{f}\bar{\Gamma}^{*k}_{[z}\bar{\Gamma}^{*f}_{w]} - \dot{\partial}_{\ell}\bar{\Gamma}^{*k}_{[z}\bar{\Gamma}^{*\ell}_{w]})),$$

$$\bar{P}_{x\ zw}^{\ y} = L\dot{\partial}_{w}\bar{\Gamma}^{*y}_{xz} - \bar{A}_{x\ w|z}^{\ y} + \bar{A}_{x}^{\ y}{}_{d}[(\dot{\partial}_{w}\bar{\Gamma}^{*d}_{bz})\dot{u}^{b} + (\dot{\partial}_{w}\bar{\Gamma}^{*d}_{n\ z})\dot{v}^{n}] + \tag{2.6}$$

$$\bar{A}_{x\ n}^{\ y}[(\dot{\partial}_{w}\bar{\Gamma}^{*n}_{bz})\dot{u}^{b} + (\dot{\partial}_{w}\bar{\Gamma}^{*n}_{mz})\dot{v}^{m}],$$

$$\bar{S}_{x\ zw}^{\ y} = 2(L\dot{\partial}_{[w}\bar{A}_{|x|z]}^{\ y} + \bar{A}_{x}^{\ d}{}_{[z}\bar{A}_{|d|\ w]}^{\ y} + \bar{A}_{x}^{\ m}{}_{[z}\bar{A}_{|m|w]}^{\ y}) \tag{2.7}$$

and each of the indices x,y,z,w belong to one of the sets $\{a,b,c,d,e,\ldots\}$ or $\{k,l,m,n,\ldots\}$. $\mathcal{D}B^{\alpha}_{a}$ and $\mathcal{D}N^{\alpha}_{k}$ are infinitesimals of a higher order and have the form

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$$(2.8)\qquad \mathcal{D}B_a^\alpha = \bar{\Theta}_a^{\ e}(d,\delta)B_e^\alpha + \bar{\Theta}_a^{\ m}(d,\delta)N_m^\alpha ,$$

$$(2.9)\qquad \mathcal{D}N_p^\alpha = \bar{\Theta}_p^{\ e}(d,\delta)B_e^\alpha + \bar{\Theta}_p^{\ m}(d,\delta)N_m^\alpha ,$$

$$(2.10)\qquad \bar{\Theta}_y^{\ x}(d,\delta) = \left[\bar{\Gamma}^{*x}_{y\ b} + L^{-1}(\bar{A}_y^{\ x}{}_c\ \bar{\Gamma}^{*c}_b + \bar{A}_y^{\ x}{}_k\ \bar{\Gamma}^{*k}_b)\right](\delta d - d\delta)u^b +$$

$$\left[\bar{\Gamma}^{*x}_{y\ k} + L^{-1}(\bar{A}_y^{\ x}{}_b\ \bar{\Gamma}^{*b}_k + \bar{A}_y^{\ x}{}_n\ \bar{\Gamma}^{*n}_k)\right](\delta d - d\delta)v^k +$$

$$\bar{A}_y^{\ x}{}_b(\delta d - d\delta)\ell^b + \bar{A}_y^{\ x}{}_n(\delta d - d\delta)\ell^n ,$$

$$x \in \{e, m\} , \qquad y \in \{a, p\} .$$

Introducing the notation

$$(2.11)\qquad \bar{K}_x^{\ y}{}_{zw} = 2\,(\partial_{[w}\bar{\Gamma}^{*y}_{|x|z]} - \dot{\partial}_d\bar{\Gamma}^{*y}_{x[z}\bar{\Gamma}^{*d}_{w]} - \dot{\partial}_\ell\bar{\Gamma}^{*y}_{x[z}\bar{\Gamma}^{*\ell}_{w]} +$$

$$+ \bar{\Gamma}^{*d}_{x[z}\bar{\Gamma}^{*y}_{|d|w]} + \bar{\Gamma}^{*\ell}_{x[z}\bar{\Gamma}^{*y}_{\ell w]})$$

after some calculation, we obtain

$$(2.12)\qquad L\bar{K}_o^{\ y}{}_{zw} \overset{def}{=} \bar{K}_a^{\ y}{}_{zw}\,\dot{u}^a + \bar{K}_k^{\ y}{}_{zw}\,\dot{v}^k = 2\,(\partial_{[w}\bar{\Gamma}^{*y}_{z]} -$$

$$- \dot{\partial}_f\bar{\Gamma}^{*y}_{[z}\Gamma^{*f}_{w]} - \dot{\partial}_\ell\bar{\Gamma}^{*y}_{[z}\bar{\Gamma}^{*\ell}_{w]}).$$

So (2.5) has the form

$$(2.13)\qquad \bar{R}_x^{\ y}{}_{zw} = \bar{K}_x^{\ y}{}_{zw} + \bar{A}_x^{\ y}{}_d\,\bar{K}_o^{\ d}{}_{zw} + \bar{A}_x^{\ y}{}_k\,\bar{K}_o^{\ k}{}_{zw}$$

THEOREM 2.1. $[\bar{\Delta}\bar{D}]\xi^a$ and $[\bar{\Delta}\bar{D}]\xi^k$ are the components of $[\Delta D]\xi^\alpha$ in the direction B_a^α and N_k^α respectively. i.e.

$$(2.14)\qquad [\Delta D]\xi^\alpha = \bar{B}_a^\alpha\,[\bar{\Delta}\bar{D}]\xi^a + N_k^\alpha\,[\bar{\Delta}\bar{D}]\xi^k .$$

PROOF. Substituting (2.2), (2.3), (2.9) and (2.10) into (2.1), we obtain

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$$[\Delta D]\xi^{\alpha} = [\bar{\Omega}_a{}^e(d,\delta)\xi^a + \bar{\Omega}_k{}^e(d,\delta)\xi^k]\ B_e^{\alpha} +$$

(2.15)

$$[\bar{\Omega}_a{}^p(d,\delta)\xi^a + \bar{\Omega}_k{}^p(d,\delta)\xi^k]\ N_p^{\alpha} + \mathcal{D}\xi^{\alpha}$$

where

$$\mathcal{D}\xi^{\alpha} = ([\delta d]\xi^e + \bar{\Theta}_a{}^e(d,\delta)\xi^a + \bar{\Theta}_k{}^e(d,\delta)\xi^k)\ B_e^{\alpha} +$$

(2.16)

$$([\delta d]\xi^m + \bar{\Theta}_a{}^m(d,\delta)\xi^a + \bar{\Theta}_k{}^m(d,\delta)\xi^k)\ N_m^{\alpha}\ .$$

On the other hand, starting from (1.23) we have

$$[\bar{\Delta}\bar{D}]\xi^a = \delta(\bar{D}\xi^a) + \bar{W}_c{}^a(\delta)\bar{D}\xi^c + \bar{W}_k{}^a(\delta)\bar{D}\xi^k - d/\delta =$$

$$(\delta\bar{W}_b{}^a(d) + \bar{W}_c{}^a(\delta)\bar{W}_b^c(d) + \bar{W}_n{}^a(\delta)\bar{W}_b{}^n(d) - d/\delta)\xi^b +$$

$$(\delta\bar{W}_k{}^a(d) + \bar{W}_c{}^a(\delta)\bar{W}_k^c(d) + \bar{W}_n^a(\delta)\bar{W}_k{}^n(d) - d/\delta)\xi^k +$$

$$[\delta d]\xi^a,$$

i.e.

$$[\bar{\Delta}\bar{D}]\xi^a = \bar{\Omega}_b{}^a(d,\delta)\xi^b + \bar{\Omega}_k{}^a(d,\delta)\xi^k + \mathcal{D}\xi^a\ , \tag{2.17}$$

where

$$\mathcal{D}\xi^a = [\delta d]\xi^a + \bar{\Theta}_b{}^a(d,\delta)\xi^b + \bar{\Theta}_k{}^a(d,\delta)\xi^k\ . \tag{2.18}$$

In a similar way we obtain

$$[\bar{\Delta}\bar{D}]\xi^k = \bar{\Omega}_b{}^k(d,\delta)\xi^b + \bar{\Omega}_m{}^k(d,\delta)\xi^m + \mathcal{D}\xi^k\ , \tag{2.19}$$

where

$$\mathcal{D}\xi^k = [\delta d]\xi^k + \bar{\Theta}_b{}^k(d,\delta)\xi^b + \bar{\Theta}_m{}^k(d,\delta)\xi^m\ . \tag{2.20}$$

Substituting (2.18) and (2.20) into (2.16), we get

$$\mathcal{D}\xi^{\alpha} = B_a^{\alpha}\mathcal{D}\xi^a + N_k^{\alpha}\mathcal{D}\xi^k. \tag{2.21}$$

Substituting (2.17), (2.19) and (2.21) into (2.15), we have (2.14) which proves Theorem 2.1.

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THEOREM 2.2. $[\bar{\Delta}\bar{D}]\xi_a$ and $[\bar{\Delta}\bar{D}]\xi_k$ are the components of $[\Delta D]\xi_\alpha$ in the direction of B^a_α and N^k_α, respectively.

PROOF. For the covariant vector field we should have DB^a_α and DN^k_α. It may be proved that

$$(2.22)\qquad DB^a_\alpha = -\bar{W}_b{}^a(d)B_\alpha{}^b - \bar{W}_k{}^a(d)\,N^k_\alpha$$

$$(2.23)\qquad DN^k_\alpha = -\bar{W}_b{}^k(d)B^b_\alpha - \bar{W}_\ell{}^k(d)N_\alpha{}^\ell\ ,$$

and these formulae are consistent with (1.1) and (1.7).

If $\bar{D}\xi_a$ and $\bar{D}\xi_k$ are defined by

$$(2.24)\qquad D\xi_\alpha = B^a_\alpha\,\bar{D}\,\xi_a + N^k_\alpha\,\bar{D}\xi_k\ ,$$

then

$$(2.25)\qquad \bar{D}\xi_a = d\xi_a - \bar{W}_a{}^b(d)\xi_b - \bar{W}_a{}^k(d)\xi_k$$

$$(2.26)\qquad \bar{D}\xi_k = d\xi_k - \bar{W}_k{}^b(d)\xi_b - \bar{W}_k{}^\ell(d)\xi_\ell .$$

In a similar way as in Theorem 2.1, we obtain

$$(2.27)\qquad [\Delta D]\xi_\alpha = B^a_\alpha\,[\bar{\Delta}\bar{D}]\xi_a + N^k_\alpha\,[\bar{\Delta}\bar{D}]\xi_k ,$$

where

$$(2.28)\qquad [\bar{\Delta}\bar{D}]\xi_a = -\bar{\Omega}_a{}^b(d,\delta)\xi_b - \bar{\Omega}_a{}^k(d,\delta)\xi_k\ ,$$

$$(2.29)\qquad [\bar{\Delta}\bar{D}]\xi_k = -\bar{\Omega}_k{}^b(d,\delta)\xi_b - \bar{\Omega}_k{}^\ell(d,\delta)\xi_\ell$$

3. ALTERNATED DIFFERENTIALS EXPRESSED BY COVARIANT DERIVATIONS

Starting from [5],

$$(3.1)\qquad \bar{D}\xi^a = \xi^a{}_{|b}\,du^b + \xi^a|_b\bar{D}\ell^b + \xi^a{}_{|m}dv^m + \xi^a|_m\,\bar{D}\ell^m ,$$

$$(3.2)\qquad \bar{D}\xi^k = \xi^k{}_{|b}\,du^b + \xi^k|_b\bar{D}\ell^b + \xi^k{}_{|m}dv^m + \xi^k|_m\bar{D}\ell^m$$

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and using

$$\bar{D}\xi^a = B^a_\alpha\, D\xi^\alpha = B^a_\alpha\,(\xi^\alpha{}_{|\beta}\, dx^\beta + \xi^\alpha|_\beta\, D\ell^\beta),$$

(3.3)

$$\bar{D}\xi^k = N^k_\alpha\, D\xi^\alpha = N^k_\alpha\,(\xi^\alpha{}_{|\beta}\, dx^\beta + \xi^\alpha|_\beta\, D\ell^\beta).$$

Further, by (1.2),(1.3), we have

(3.4) $$B^\beta_b\, \xi^\alpha{}_{|\beta} = B^\alpha_a\, \xi^a{}_{|b} + N^\alpha_m\, \xi^m{}_{|b},$$

(3.5) $$N^\beta_k\, \xi^\alpha{}_{|\beta} = B^\alpha_a\, \xi^a{}_{|k} + N^\alpha_m\, \xi^m{}_{|k}$$

(3.6) $$B^\beta_b\, \xi^\alpha|_\beta = B^\alpha_a\, \xi^a|_b + N^\alpha_m\, \xi^m|_b,$$

(3.8) $$N^\beta_k\, \xi^\alpha|_\beta = B^\alpha_a\, \xi^a|_k + N^\alpha_m\, \xi^m|_k.$$

Using (1.7) and (3.4) - (3.7), we get
and (3.4)-(3.7), we get

(3.8) $$\xi^\alpha|_\delta = B^a_\delta\,(B^\alpha_b\, \xi^b{}_{|a} + N^\alpha_k\, \xi^k{}_{|a}) +$$

$$N^k_\delta\,(B^\alpha_a\, \xi^a{}_{|k} + N^\alpha_m\, \xi^m{}_{|k}).$$

(3.9) $$\xi^\alpha|_\delta = B^a_\delta\,(B^\alpha_b\, \xi^b|_a + N^\alpha_k\, \xi^k|_a) +$$

$$N^k_\delta\,(B^\alpha_b\, \xi^b|_k + N^\alpha_m\, \xi^m|_k).$$

From (3.3)-(3.6) or (3.8) and (3.9), we can easily obtain

$$\xi^a{}_{|b} = \xi^\alpha{}_{|\beta}\, B^a_\alpha\, B^\beta_b$$

(3.10) $$\xi^m{}_{|b} = \xi^\alpha{}_{|\beta}\, N^m_\alpha\, B^\beta_b$$

$$\vdots$$

$$\xi^m|_k = \xi^\alpha|_\beta\, N^m_\alpha\, N^\beta_k\ .$$

In the same way for the covariant vector field ξ_α, we have

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$$\xi_{a|b} = \xi_{\alpha|\beta}\, B_a^{\alpha}\, {}_b^{\beta}$$

$$\xi_{m|b} = \xi_{\alpha|\beta}\, N_m^{\alpha}\, B_b^{\beta} \tag{3.11}$$

$$\vdots$$

$$\xi_m|_k = \xi_\alpha|_\beta\, N_m^{\alpha}\, N_k^{\beta}$$

We shall express $[\bar{\Delta}\bar{D}]\ \xi^a$ and $[\bar{\Delta}\bar{D}]\ \xi^k$ using the covariant derivations. Starting from (3.1), we have

$$\bar{\Delta}\bar{D}\xi^a = \xi^a{}_{|b|c}\, du^b \delta u^c + \xi^a|_{b|c}\, \bar{D}\ell^b \delta u^c + \xi^a{}_{|m|c}\, dv^m\, \delta u^c + \xi^a|_{m|c}\, \bar{D}\ell^m\, \delta u^c +$$

$$\xi^a{}_{|b|k}\, du^b \delta v^k + \xi^a|_{b|k}\, \bar{D}\ell^b \delta v^k + \xi^a{}_{|m|k}\, dv^m \delta v^k + \xi^a|_{m|k}\, \bar{D}\ell^m \delta v^k +$$

$$\xi^a{}_{|b}|_c\, du^b \bar{\Delta}\ell^c + \xi^a|_b{}_c\, \bar{D}\ell^b \bar{\Delta}\ell^c + \xi^a{}_{|m}|_c\, dv^m \bar{\Delta}\ell^c + \xi^a|_m{}_c\, \bar{D}\ell^m \bar{\Delta}\ell^c +$$

$$\xi^a{}_{|b|k}\, du^b\, \bar{\Delta}\ell^k + \xi^a|_b{}_k\, \bar{D}\ell^b \bar{\Delta}\ell^k + \xi^a{}_{|m}|_k\, dv^m\, \bar{\Delta}\ell^k + \xi^a|_m{}_k\, \bar{D}\ell^m \bar{\Delta}\ell^k +$$

$$\xi^a{}_{|n}\, \bar{\Delta}\bar{D}\, u^b + \xi^a|_b\, \bar{\Delta}\bar{D}\ell^b + \xi^a{}_{|m}\, \bar{\Delta}\bar{D}v^m + \xi^a|_m\, \bar{\Delta}\bar{D}\ell^m\,,$$

from which we can get, after some calculation,

$$[\bar{\Delta}\bar{D}]\ \xi^a = \xi^a_{[|b|c]}\, [du^b\, \delta u^c] + 2\, \xi^a_{[|b}|_{c]}\, [du^b\, \bar{\Delta}\ell^c] +$$

$$2\xi^a_{[|b|k]}\, [du^b\, \delta v^k] + 2\xi^a_{[|b|k]}\, [du^b\, \bar{\Delta}\ell^k] +$$

$$2\xi^a_{[|k|b]}\, [dv^k\, \bar{\Delta}\ell^b] + \xi^a_{[|k|\ell]}\, [dv^k\, \delta v^\ell] + \tag{3.12}$$

$$2\xi^a_{[|k|\ell]}\, [dv^k\, \bar{\Delta}\ell^\ell] + \xi^a_{[|b|c]}\, [\bar{D}\ell^b\, \bar{\Delta}\ell^c] +$$

$$2\xi^a_{[|b|k]}\, [\bar{D}\ell^b\, \bar{\Delta}\ell^k] + \xi^a_{[|k|\ell]}\, [\bar{D}\ell^k\, \bar{\Delta}\ell^\ell] + B^a,$$

where

$$B^a = \xi^a{}_{|d}[\bar{\Delta}\bar{D}]\, u^d + \xi^a|_d[\bar{\Delta}\bar{D}]\, \ell^d + \xi^a{}_{|k}\, [\bar{\Delta}\bar{D}]\, v^k + \xi^a|_k\, [\bar{\Delta}\bar{D}]\, \ell^k\,. \tag{3.13}$$

We shall first calculate B^a. As

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$$\xi^a{}_{|d}\left[\bar{\Delta}\bar{D}\right] u^d = \xi^a{}_{|d}\,\Delta du^d - d/\delta =$$

$$\{\xi^a{}_{|d}\left[\delta du^d + (\bar{\Gamma}^{*d}_{b\,c}\,\delta u^c + \bar{\Gamma}^{*d}_{b\,k}\,\delta v^k + \bar{A}_b{}^d{}_c\,\bar{\Delta}\ell^c + \bar{A}_b{}^d{}_k\bar{\Delta}\ell^k).du^b\right.$$

$$\left. + (\bar{\Gamma}^{*d}_{k\,c}\,\delta u^c + \bar{\Gamma}^{*d}_{k\,\ell}\delta v^\ell + \bar{A}_k{}^d{}_c\,\bar{\Delta}\ell^c + \bar{A}_k{}^d{}_\ell\bar{\Delta}\ell^\ell)dv^k\right]\}-\{d/\delta\}.$$

$$\xi^a{}_{|k}\left[\bar{\Delta}\bar{D}\right]v^k = \{\xi^a{}_{|k}\left[\delta dv^k + \bar{w}_b{}^k(\delta)du^b + \bar{w}_\ell{}^k(\delta)dv^\ell\right]\}-\{d/\delta\},$$

$$\xi^a|_b\,\left[\bar{\Delta}\bar{D}\right]\ell^b = \{\xi^a|_b\left[\delta\bar{D}\ell^b + \bar{w}_c{}^b(\delta)\bar{D}\ell^c + \bar{w}_k{}^b(\delta)\bar{D}\ell^k\right]\}-\{d/\delta\},$$

$$\xi^a|_m\left[\bar{\Delta}\bar{D}\right]\ell^m = \{\xi^a|_m\,\left[\delta\bar{D}\ell^m + \bar{w}_c{}^m(\delta)\bar{D}\ell^c + \bar{w}_k{}^m(\delta)\bar{D}\ell^k\right]\}-\{d/\delta\},$$

and using the evaluated expressions for $\delta\bar{D}\ell^b$, $\delta\bar{D}\ell^m$ from [5] we have

$$B^a = \left[\xi^a{}_{|d}\,\bar{\Gamma}^{*\,d}_{[b\,c]} + \xi^a{}_{|m}\,\bar{\Gamma}^{*\,m}_{[b\,c]} + \xi^a|_d\,\tfrac{1}{2}\,\bar{K}_o{}^d{}_{bc} + \tfrac{1}{2}\,\xi^a|_m\,\bar{K}_o{}^m{}_{b\,c}\right]\left[du^b\delta u^c\right] +$$

$$+ \left[2(\xi^a{}_{|d}\,\bar{\Gamma}^{*\,d}_{[b\,k]} + \xi^a|_m\,\bar{\Gamma}^{*\,m}_{[b\,k]}) + \xi^a|_d\,\bar{K}_o{}^d{}_{b\,k} + \xi^a|_m\,\bar{K}_o{}^m{}_{b\,k}\right]\left[du^b\,\delta v^k\right] +$$

$$\left[\xi^a{}_{|d}\,\Gamma^{*\,d}_{[k\ell]} + \xi^a{}_{|m}\,\Gamma^{*\,m}_{[k\,\ell]} + \tfrac{1}{2}\,\xi^a|_d\,\bar{K}_o{}^d{}_\ell + \tfrac{1}{2}\,\xi^a|_m\,\bar{K}_o{}^m{}_{k\ell}\right]\left[dv^k\delta v^\ell\right] +$$

$$\left[\xi^a{}_{|d}\,\bar{A}_b{}^d{}_c + \xi^a{}_{|m}\,\bar{A}_b{}^m{}_c - \xi^a|_d\,\bar{\Gamma}^{*\,d}_{c\,b} - \xi^a|_m\,\bar{\Gamma}^{*\,m}_{c\,b} + \xi^a|_d\,\dot{\partial}_c\,\bar{\Gamma}^{*\,d}_{b} +\right.$$

$$\left. + \xi^a|_k\,\dot{\partial}_c\,\Gamma^{*\,k}_{b}\right]\cdot\left[du^b\,\bar{\Delta}\ell^c\right] +$$

(3.14) $$\left[\xi^a{}_{|d}\,\bar{A}_b{}^d{}_k + \xi^a{}_{|m}\,\bar{A}_b{}^m{}_k + \xi^a|_d(\dot{\partial}_k\bar{\Gamma}^{*d}_b - \bar{\Gamma}^{*\,d}_{k\,b}) + \xi^a|_m\,(\dot{\partial}_k\,\bar{\Gamma}^{*m}_b -\right.$$

$$\left. - \bar{\Gamma}^{*\,m}_{k\,b})\right]\left[du^b\,\bar{\Delta}\ell^k\right] +$$

$$\left[\xi^a{}_{|d}\,\bar{A}_k{}^d{}_b + \xi^a{}_{|m}\,\bar{A}_k{}^m{}_b + \xi^a|_d(\dot{\partial}_b\,\bar{\Gamma}^{*d}_k - \bar{\Gamma}^{*\,d}_{b\,k}) + \xi^a|_m\,(\dot{\partial}_b\bar{\Gamma}^{*m}_k -\right.$$

$$\left. - \bar{\Gamma}^{*m}_{b\,k})\right]\left[dv^k\bar{\Delta}\ell^b\right] +$$

$$\left[\xi^a{}_{|d}\,\bar{A}_k{}^d{}_\ell + \xi^a{}_{|m}\,\bar{A}_k{}^m{}_\ell + \xi^a|_d\,(\dot{\partial}_\ell\,\Gamma^{*d}_{k} - \bar{\Gamma}^{*\,d}_{\ell\,k}) + \xi^a|_m\,(\dot{\partial}_\ell\,\Gamma^{*m}_{k} - \bar{\Gamma}^{*\,m}_{\ell\,k})\right]\left[dv^k\bar{\Delta}\ell^\ell\right]$$

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$$\left[\xi^a|_d \bar{A}_{[b\ c]}^{\ d} + \xi^a|_m \bar{A}_{[b\ c]}^{\ m}\right]\left[\bar{D}\ell^b \bar{\Delta}\ell^c\right] + \left[\xi^a|_d \bar{A}_{b\ k}^{\ d} + \xi^a|_m \bar{A}_{b\ k}^{\ m}\right]\left[\bar{D}\ell^b \bar{\Delta}\ell^k\right] +$$

$$\left[\xi^a|_d \bar{A}_{[k\ \ell]}^{\ d} + \xi^a|_m \bar{A}_{[k\ \ell]}^{\ m}\right]\left[\bar{D}\ell^k \bar{\Delta}\ell^\ell\right] + \mathcal{D}_1 \xi^a ,$$

where

$$\mathcal{D}_1 \xi^a = \left[\xi^a{}_{|b} + \xi^a|_d L^{-1}\bar{\Gamma}^{*d}_b + \xi^a|_m L^{-1}\bar{\Gamma}^{*m}_b\right][\delta d]u^b +$$

$$\left[\xi^a{}_{|k} + \xi^a|_d L^{-1}\bar{\Gamma}^{*d}_k + \xi^a|_m L^{-1}\bar{\Gamma}^{*m}_k\right][\delta d]v^k + \tag{3.15}$$

$$\xi^a|_b [\delta d]\ell^b + \xi^a|_k [\delta d]\ell^k .$$

Writing the coefficient explicitly beside $[\delta d]u^b$ in (3.15) we get the expression

$$\partial_b \xi^a - \dot{\partial}_d \xi^a \bar{\Gamma}^{*d}_b - \dot{\partial}_k \xi^a \bar{\Gamma}^{*k}_b + \bar{\Gamma}^{*a}_{c\ b}\xi^c + \bar{\Gamma}^{*a}_{k\ b}\xi^k + L^{-1}(\dot{\partial}_d \zeta^a + \bar{A}^{\ a}_{c\ d}\xi^c + \bar{A}^{\ a}_{k\ d}\xi^k)\bar{\Gamma}^{*d}_b +$$

$$+ L^{-1}(\dot{\partial}_k \xi^a + \bar{A}^{\ a}_{c\ k}\xi^c + \bar{A}^{\ a}_{\ell\ k}\xi^\ell)\Gamma^{*k}_b ,$$

which is the same as the coefficient beside $[\delta d]\ u^b$ in (2.17) because we have

$$[\delta d]\xi^a = \partial_b \xi^a\ [\delta d]u^b + \partial_k \xi^a [\delta d]v^k + L\dot{\partial}_b\ \xi^a [\delta d]\ell^b + L\dot{\partial}_n\ \xi^a [\delta d]\ell^n +$$

$$(\ell^b\ \dot{\partial}_b\ \xi^a + \ell^n\ \dot{\partial}_n\ \xi^a)\ [\delta d]L$$

under condition that ξ^a is homogeneous of degree zero in u^b and v^k i.e.

$$\ell^b \dot{\partial}_b\ \xi^a + \ell^n \dot{\partial}_n \xi^a = 0. \tag{3.16}$$

Comparing the coefficients of $[\delta d]v^k$, $[\bar{\Delta}\bar{D}]\ell^b$ and $[\bar{\Delta}\bar{D}]\ell^n$ in (3.15) and (2.17), we get

$$\mathcal{D}\xi^a = \mathcal{D}_1\ \xi^a \tag{3.17}$$

From (3.12)(3.14) and (2.16), we get

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$$(3.18)\qquad \xi^a_{[|x|y]} + \xi^a_{|d}\,\bar{\Gamma}^{*\,d}_{[x\,y]} + \xi^a_{|m}\,\bar{\Gamma}^{*\,m}_{[x\,y]} = \frac{1}{2}\,(\bar{R}^a_{d\,xy}\,\xi^d + \bar{R}^a_{m\,xy}\,\xi^m +$$

$$\xi^a|_d\,\bar{K}^{\,d}_{o\,xy} + \xi^a|_m\,\bar{K}^{\,m}_{o\,xy}),$$

$$(3.19)\qquad 2\xi^a_{[|x|_y]} + \xi^a_{|d}\,\bar{A}^{\,d}_{x\,y} + \xi^a_{|m}\,\bar{A}^{\,m}_{x\,y} + \xi^a|_d\,(\dot{\partial}_y\,\bar{\Gamma}^{*d}_{x} - \bar{\Gamma}^{*\,d}_{y\,x})$$

$$+ \xi^a|_m\,(\dot{\partial}_y\,\bar{\Gamma}^{*m}_{x} - \bar{\Gamma}^{*}_{yx}) = \bar{P}^{\,a}_{d\,xy}\,\xi^d + \bar{P}^{\,a}_{m\,xy}\,\xi^m,$$

$$(3.20)\qquad \xi^a_{[|x|_y]} + \xi^a|_d\,\bar{A}^{\,d}_{[x\,y]} + \xi^a|_m\,\bar{A}^{\,m}_{[x\,y]} = \frac{1}{2}\,\bar{S}^{\,d}_{a\,xy}\,\xi^a + \frac{1}{2}\,S^{\,a}_{m\,xy}\,\xi^m,$$

$$x \in \{b, k\} \qquad y \in \{c, \ell\}$$

Relations (3.18)-(3.20) are valid if index a is substituted in them by $p\,(p = m+1, \ldots n)$.

4. DOUBLE ALTERNATED DIFFERENTIALS OF CURVATURE TENSORS

In [5] we obtained the relations which connect the curvature tensors of the Finsler space F_n and its subspaces F_m and F_{n-m}. These formulae have the form

$$\bar{R}_{a\,d\,b\,c} = R_{\delta\kappa\beta\gamma}\,B^{\delta\ \kappa\ \beta\ \gamma}_{a\ d\ b\ c}$$

$$\bar{R}_{a\,d\,b\,k} = R_{\delta\kappa\beta\gamma}\,B^{\delta\ \kappa\ \beta\ \gamma}_{a\ d\ b\ c}$$

$$\vdots$$

$$R_{k\,\ell\,m\,n} = R_{\delta\kappa\beta\gamma}\,N^{\delta\ \kappa\ \beta\ \gamma}_{k\ \ell\ m\ n}$$

and valid for the tensors P and S. Using the relation

$$(4.2)\qquad \delta^{\alpha}_{\beta} = B^{\alpha\ a}_{a\ \beta} + N^{\alpha}_{k}\,N^{k}_{\beta}$$

several times, we obtain

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$$
\begin{aligned}
R_{\alpha\beta\gamma\delta} = {} & \bar{R}_{abcd} B^{a\,b\,c\,d}_{\alpha\,\beta\,\gamma\,\delta} + \bar{R}_{abcn} B^{a\,b\,c}_{\alpha\,\beta\,\gamma} N^{n}_{\delta} + \bar{R}_{abmd} B^{a\,b}_{\alpha\,\beta} N^{m}_{\gamma} B^{d}_{\delta} + \\
& \bar{R}_{a\ell cd} B^{a}_{\alpha} N^{\ell}_{\beta} B^{c\,d}_{\gamma\,\delta} + \bar{R}_{kbcd} N^{k}_{\alpha} B^{b\,c\,d}_{\beta\,\gamma\,\delta} + \bar{R}_{abmn} B^{a\,b}_{\alpha\,\beta} N^{m\,n}_{\gamma\,\delta} + \\
& \bar{R}_{a\ell cn} B^{a}_{\alpha} N^{\ell}_{\beta} B^{c}_{\gamma} N^{n}_{\delta} + \bar{R}_{kbcn} N^{k}_{\alpha} B^{b\,c}_{\beta\,\gamma} N^{n}_{\delta} + \bar{R}_{a\ell md} B^{a}_{\alpha} N^{\ell\,m}_{\beta\,\gamma} B^{d}_{\delta} + \\
(4.3) \quad & \bar{R}_{kbnd} N^{k}_{\alpha} B^{b}_{\beta} N^{n}_{\gamma} B^{d}_{\delta} + \bar{R}_{k\ell cd} N^{k\,\ell}_{\alpha\,\beta} B^{c\,d}_{\gamma\,\delta} + \bar{R}_{a\ell mn} B^{a}_{\alpha} N^{\ell\,m\,n}_{\beta\,\gamma\,\delta} + \\
& \bar{R}_{kbmn} N^{k}_{\alpha} B^{b}_{\beta} N^{m\,n}_{\gamma\,\delta} + \bar{R}_{k\ell cn} N^{k\,\ell}_{\alpha\,\beta} B^{c}_{\gamma} N^{n}_{\delta} + \bar{R}_{k\ell md} N^{k\,\ell\,m}_{\alpha\,\beta\,\gamma} B^{d}_{\delta} + \\
& R_{k\ell mn} N^{k\,\ell\,m\,n}_{\alpha\,\beta\,\gamma\,\delta}
\end{aligned}
$$

Relation (4.3) is valid for the tensors P and S also.

If the absolute differentials D_1 and D_2 are defined in a similar fashion as D and Δ, we obtain from (4.3).

$$
\begin{aligned}
(4.4) \quad [D_1D_2]\, R_{\alpha\beta\gamma\delta} = {} & \bar{R}_{abcd} ([D_1D_2]B^{a}_{\alpha})\, B^{b\,c\,d}_{\beta\,\gamma\,\delta} + \\
& \bar{R}_{abcd} B^{a}_{\alpha} ([D_1D_2]B^{b}_{\beta}) B^{c\,d}_{\gamma\,\delta} + \ldots\ldots\ldots + \\
& \bar{R}_{kbcd} ([D_1D_2]N^{k}_{\alpha})\, B^{b\,c\,d}_{\beta\,\gamma\,\delta} + \ldots\ldots + \bar{R}_{k\ell mn} N^{k\,\ell\,m}_{\alpha\,\beta\,\gamma} [D_1D_2] N^{n}_{\delta}
\end{aligned}
$$

There are $16.4 = 64$ summands of the right hand side of (4.4). Using relation (4.1), we have

$$
\begin{aligned}
& B^{\alpha}_{a}\, [D_1D_2]\, B^{a}_{\beta} + N^{\alpha}_{k}\, [D_1D_2]\, N^{k}_{\beta} = \\
(4.5) \quad & -([D_1D_2]\, B^{\alpha}_{a})\, B^{a}_{\alpha} - ([D_1D_2] N^{\alpha}_{k})\, N^{k}_{\beta}
\end{aligned}
$$

Using (4.5), (2.2), (2.3) and summing the terms which appear in (4.4), we obtain

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$$\bar{R}_{a\,b\,c\,d}\,([D_1D_2]B^a_\alpha)B^{b\;c\;d}_{\beta\;\gamma\;\delta} + \bar{R}_{k\,b\,c\,d}\,([D_1D_2]N^k_\alpha)\,B^{b\;c\;d}_{\beta\;\gamma\;\delta} =$$

$$\bar{R}_{\varepsilon\,b\,c\,d}\,B^{b\;c\;d}_{\beta\;\gamma\;\delta}\,(B^\varepsilon_a\,[D_1D_2]\,B^a_\alpha + N^\varepsilon_k\,[D_1D_2]\,N^k_\alpha) =$$

$$\bar{R}_{\varepsilon\,b\,c\,d}\,B^{b\;c\;d}_{\beta\;\gamma\;\delta}\,(-([D_1D_2]B^\varepsilon_a)\,B^a_\alpha - ([D_1D_2]\,N^\varepsilon_k)\,N^k_\alpha) =$$

$$\bar{R}_{\varepsilon\,b\,c\,d}\,B^{b\;c\;d}_{\beta\;\gamma\;\delta}\,(-\bar{\Omega}_a{}^e\,(d_2d_1)\,B^\varepsilon_e - \bar{\Omega}_a{}^p(d_2d_1)\,N^\varepsilon_p)\,B^a_\alpha +$$

$$\bar{R}_{\varepsilon\,b\,c\,d}\,B^{b\;c\;d}_{\beta\;\gamma\;\delta}\,(-\bar{\Omega}_k{}^e(d_2d_1)\,B^\varepsilon_e - \bar{\Omega}_k{}^p\,(d_2d_1)\,N^\varepsilon_p)\,N^k_\alpha =$$

$$-\bar{R}_{e\,b\,c\,d}\,\bar{\Omega}_a{}^e\,(d_2d_1)\,B^{a\;b\;c\;d}_{\alpha\;\beta\;\gamma\;\delta} - \bar{R}_{p\,b\,c\,d}\,\bar{\Omega}_a{}^p(d_2d_1)\,B^{a\;b\;c\;d}_{\alpha\;\beta\;\gamma\;\delta} -$$

$$-\bar{R}_{e\,b\,c\,d}\,\bar{\Omega}_k{}^e\,(d_2d_1)\,N^k_\alpha\,B^{b\;c\;d}_{\beta\;\gamma\;\delta} - \bar{R}_{p\,b\,c\,d}\,\bar{\Omega}_k{}^p\,(d_2d_1)\,N^k_\alpha\,B^{b\;c\;d}_{\beta\;\gamma\;\delta}.$$

There are 16.2 pair of summands on the right hand side of (4.4) which give by using the above method 16.2.4 summands. Since

$$\begin{aligned}\left[\bar{D}_1\bar{D}_2\right]\bar{R}_{x\,y\,u\,w} = &-\bar{R}_{e\,y\,u\,w}\,\bar{\Omega}_x{}^e\,(d_2d_1) - \bar{R}_{p\,y\,u\,w}\,\bar{\Omega}_x{}^p\,(d_2d_1)\\ &-\bar{R}_{x\,e\,y\,w}\,\bar{\Omega}_y{}^e(d_2d_1) - \bar{R}_{x\,p\,u\,w}\,\bar{\Omega}_y{}^p(d_2d_1) -\\ &-\bar{R}_{x\,y\,e\,w}\,\bar{\Omega}_u{}^e(d_2d_1) - \bar{R}_{x\,y\,p\,w}\,\bar{\Omega}_u{}^p(d_2d_1)\\ &-\bar{R}_{x\,y\,u\,e}\,\bar{\Omega}_w{}^e(d_2d_1) - \bar{R}_{x\,y\,u\,p}\,\bar{\Omega}_w{}^p(d_2d_1),\end{aligned} \tag{4.6}$$

where each of the indices x,y,u,w belongs to one of the set {a,b,c,d,...} or {k,ℓ,m,n,...}; (4.4) takes the form

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$$
\begin{aligned}
[D_1D_2]\, R_{\alpha\beta\gamma\delta} = {} & [\bar{D}_1\bar{D}_2]\bar{R}_{abcd}\, B^{abcd}_{\alpha\beta\gamma\delta} + [\bar{D}_1\bar{D}_2]\, R_{abcn}\, B^{abc}_{\alpha\beta\gamma} N^{n}_{\delta} \\
& [\bar{D}_1\bar{D}_2]\bar{R}_{abmd}\, B^{ab}_{\alpha\beta} N^{m}_{\gamma} B^{d}_{\delta} + [\bar{D}_1\bar{D}_2]\, \bar{R}_{a\ell cd}\, B^{a}_{\alpha} N^{\ell}_{\beta} B^{cd}_{\gamma\delta} + \\
& [\bar{D}_1\bar{D}_2]\bar{R}_{kbcd}\, N^{k}_{\alpha} B^{bcd}_{\beta\gamma\delta} + [\bar{D}_1\bar{D}_2]\, \bar{R}_{abmn}\, B^{ab}_{\alpha\beta} N^{mn}_{\gamma\delta} + \\
& [\bar{D}_1\bar{D}_2]\bar{R}_{a\ell cn}\, B^{a}_{\alpha} N^{\ell}_{\beta} B^{c}_{\gamma} N^{n}_{\delta} + [\bar{D}_1\bar{D}_2]\, \bar{R}_{kbcn}\, N^{k}_{\alpha} B^{bc}_{\beta\gamma} N^{n}_{\delta} + \\
& [\bar{D}_1\bar{D}_2]\bar{R}_{a\ell md}\, B^{a}_{\alpha} N^{\ell m}_{\beta\gamma} B^{d}_{\delta} + [\bar{D}_1\bar{D}_2]\, \bar{R}_{kbnd}\, N^{k}_{\alpha} B^{b}_{\beta} N^{n}_{\gamma} B^{d}_{\delta} \\
& [\bar{D}_1\bar{D}_2]\bar{R}_{k\ell cd}\, N^{k\ell}_{\alpha\beta} B^{cd}_{\gamma\delta} + [\bar{D}_1\bar{D}_2]\, \bar{R}_{a\ell mn}\, B^{a}_{\alpha} N^{\ell\, mn}_{\beta\ \gamma\delta} + \\
& [\bar{D}_1\bar{D}_2]\bar{R}_{kbmn}\, N^{k}_{\alpha} B^{b}_{\beta} N^{mn}_{\gamma\delta} + [\bar{D}_1\bar{D}_2]\, \bar{R}_{k\ell cn}\, N^{k\ell}_{\alpha\beta} B^{c}_{\gamma} N^{n}_{\delta} + \\
& [\bar{D}_1\bar{D}_2]\bar{R}_{k\ell md}\, N^{k\ell m}_{\alpha\beta\gamma} B^{d}_{\delta} + [\bar{D}_1\bar{D}_2]\bar{R}_{k\ell mn}\, N^{k\ell mn}_{\alpha\beta\gamma\delta}\,.
\end{aligned}
\tag{4.7}
$$

(4.7) is valid for the tensors P and S.

From (4.7), we obtain

$$
\begin{aligned}
[\bar{D}_1\bar{D}_2]\, \bar{R}_{abcd} &= ([D_1D_2]\, R_{\alpha\beta\gamma\delta})\, B^{\alpha\beta\gamma\delta}_{abcd} \\
[\bar{D}_1\bar{D}_2]\, \bar{R}_{abcn} &= ([D_1D_2]\, R_{\alpha\beta\gamma\delta})\, B^{\alpha\beta\gamma}_{abc} N^{\delta}_{n} \qquad (4.8) \\
&\ \ \vdots \\
[\bar{D}_1\bar{D}_2]\, \bar{R}_{k\ell mn} &= ([D_1D_2]\, R_{\alpha\beta\gamma\delta})\, N^{\alpha\beta\gamma\delta}_{k\ell mn}
\end{aligned}
$$

From (4.7) and (4.8), we obtain

THEOREM 4.1 The necessary and sufficient conditions that the Finsler space has a reccurent curvature tensor R of the second order i.e.

$$[D_1D_2]\, R_{\alpha\beta\gamma\delta} = K\, R_{\alpha\beta\gamma\delta} \quad \text{are} \tag{4.9}$$

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$$[\bar{D}_1\bar{D}_2]\ \bar{R}_{a\ b\ c\ d} = K\ \bar{R}_{a\ b\ c\ d}$$

(4.10)
$$[\bar{D}_1\bar{D}_2]\ \bar{R}_{a\ b\ c\ n} = K\ \bar{R}_{a\ b\ c\ n}$$

$$\vdots$$

$$[\bar{D}_1\bar{D}_2]\ \bar{R}_{k\ \ell\ m\ n} = K\ \bar{R}_{k\ \ell\ m\ n}$$

The same theorem is true if everywhere in (4.9) and (4.10) instead of tensor R we put P or S with the same indices as those of R.

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REZIME

NEKE RELACIJE KOJE ZADOVOLJAVAJU TENZORI KRIVINA FINSLEROVOG PROSTORA

Ovde se posmatraju specijalni Finslerovi prostori u kojima postoje vektorska polja $B_a^\alpha(x)$ i $N_k(x)$ za koje važi $g_{\alpha\beta}(x\dot{x})\ B_a^\alpha(x)\ N_k^\alpha(x) = 0$. Dato je razlaganje alternisanog diferencijala nekog vektorskog polja kao i tenzora krivine u pravcu B_a^α i N_k^α.

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PARALLEL COMPUTATION OF VORONOI DIAGRAMS

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ABSTRACT

The paper presents algorithms for solving some computational geometry problems on a shared memory parallel computer, where concurrent reads are allowed but no two processors can simultaneously attempt to write in the same memory location. We present new $O(\log^3(n))$ time with $O(n \log(n))$ processors algorithm for construction of the Voronoi diagrams and show that the algorithm given in [1] for solving the same problem does not work correctly. Also, we make the convex hull algorithm described in [1] more efficient by omitting their last step which is superfluous due to a too weak lemma they have given.

1. INTRODUCTION

Since they involve asking basic questions about sets of points, lines, polygons, etc, geometric problems arise often in many applications [9]. We are interested in finding parallel algorithms solving some of these problems which are efficient both in terms of their running time and in the number of processors used. In this paper we shall consider two basic geometric notions: the Voronoi diagram and the convex hull of a set of n points.

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Key words and phrases: Parallel computation, computational geometry, Voronoi diagram, convex hull.

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Throughout this paper, the computational model used is the shared memory in which concurrent reads are allowed, but no two processors should attempt to simultaneously write in the same memory location. We shall henceforth refer to this model as the CREW PRAM (concurrent read exclusive write parallel RAM).

In [9] it is pointed out that the Voronoi diagram can be used as a powerful tool to give efficient algorithms for a wide variety of other geometric problems. For example, from the Voronoi diagram we can easily obtain the closest neighbour of each site (point), or the Delaney triangulation, or the Euclidean minimum spanning tree of the n sites, or the minimal distance between two convex sets, or the largest point free circle with center inside their convex hull, et cetera.

Geometric algorithms for the CREW PRAM model of computation have been studied in literature. We shall mention two optimal parallel algorithms for computing the planar convex hull [1,3] and 3-dimensional convex hull [4,1] , an efficient algorithm for selecting the closest pair of points [3] and an algorithm for computing the minimal circumscribing triangle to a convex n-gon [1] .

2. PARALLEL COMPUTATION OF VORONOI DIAGRAMS

In [1] a $O(\log^3(n))$ time with $O(n)$ processors algorithm to compute the planar Voronoi diagram of a set of n points is presented. We show that the algorithm does not work correctly because the main problem in the merge step of the construction is not solved (or even mentioned) and give a correct algorithm.

Let S be a set of n points. Given any p in S, the (open) Voronoi cell V(p,S) consists of all the points in the plane closer to p than to any other point in S; it is well-known that this is a planar graph with O(n) straight-line edges, O(n) vertices, and the n cells are bounded by convex polygons. In nondegenerate cases each Voronoi vertex is defined by three points from S, and each Voronoi edge is the bisector of two points from S. First sequential method for computing Vor(S) is presented in [10].

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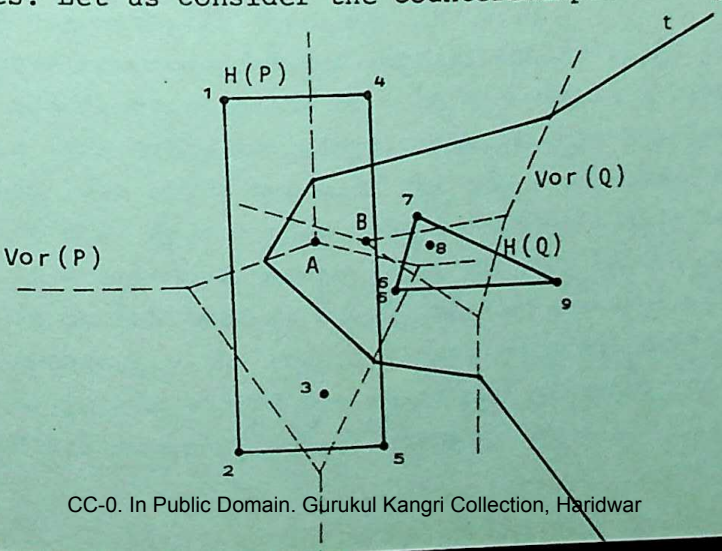
Suppose that S is partitioned into P U Q (i.e. there is a vertical line L which has all of P to its left and all of Q to its right). This can be accomplished by sorting the points in O(log(n)) time and O(n) processors [2,6]. Recursively find Vor(P) and Vor(Q). The merging of Vor(P) and Vor(Q) is defined by by a polygonal path t, called the contour, separating P from Q. The contour t cuts the plane into a left portion t(l) and a right portion t(r). Then the Voronoi diagram Vor(S) is the union of Vor(P) ∩ t(l) and Vor(Q) ∩ t(r).

The main task of a parallel algorithm is to identify those edges (and hence, those cells) of Vor(P) and Vor(Q) which intersect the contour. As claimed in [1], an edge in Vor(P) intersects the contour if and only if one end is closer to P and the other end is closer to Q, so to identify such edges it is enough to locate their endpoints in the appropriate cells of Vor(Q) and compare distances [1].

The main problem in the algorithms given in [1] is that they assume that all the vertices of Vor(P) lying inside H(P) (the convex hull of P) are certainly vertices of Vor(P U Q) (similarly for vertices of Vor(Q)). Following the assumption, they locate only the vertices of Vor(P) lying outside H(P) into sectors (see definition in [1]) of R^2-H(Q) (the complement of H(Q)) and into the correct Voronoi cell for Q if the cell is not completely inside H(Q). This enables them to use the binary tree structure of Vor(P)-H(P) to give simpler constructions for locating points. Let us consider the counterexample on Fig. 1.



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The vertex A of Vor(P) lies inside H(P), but it is not a vertex of Vor(P U Q) while vertex B of Vor(Q) lies inside H(P) but still remains as a vertex of Vor(P U Q).

Thus, it is necessary to determine for each vertex of Vor(P) the appropriate cell in Vor(Q) and similarly for each vertex of Vor(Q) the appropriate cell in Vor(P) in order to determine the edges of contour t and the endpoints of the edges.

In order to locate a point into Voronoi cells of P or Q we shall use the chain method described by Lee and Preparata in [5,9] for planar straight-line graphs. First we shall describe a data structure to support the planar location of a point into appropriate Voronoi cell of Vor(S).

We can organize all the contours that appear in the divide-and-conquer solution as nodes of a balanced binary tree (Fig. 2). Each contour is a single monotone chain (cf. [10,9]. Here a monotone chain is a chain of edges such that any horizontal line intersects it in exactly one point. Also, in the process contours will define a monotone complete set of chains (i.e. the set of chains so that for any two chains c(1) and c(2) the vertices of c(1) that are not on c(2) are on the same side of c(2)). We shall label each contour by the number

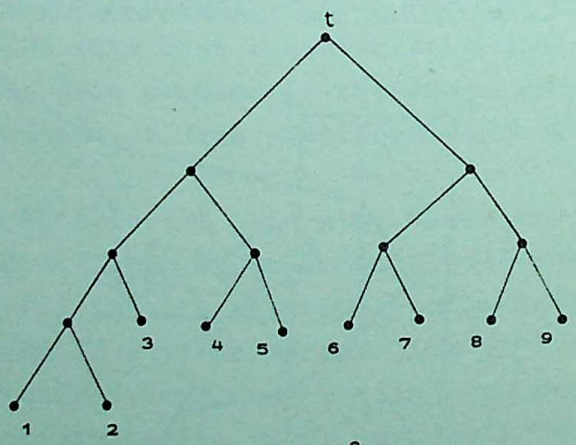

Fig. 2.

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of points from S that lie to their left. So, each Voronoi cell is situated between two neighbouring contours. In the divide--and-conquer process, each edge of a new contour is assigned to the contour. However, each new contour can intersect some of the previous ones. In this case all the intersected contours are modified by discarding their edges that are eliminated from the Voronoi diagram and including corresponding part of the new contour instead. Thus not all the edges that belong to a contour are necessarily assigned to the contour because each edge of the Voronoi diagram is assigned exactly ones at the time when it appears in a contour by a merge. Afterwords the edge can only be intersected or discarded but it cannot be assigned to another contour. If an edge e belongs to more than one Voronoi chain (i.e. contour) it belongs to all the members of a set (an interval) of consecutive chains. Each edge e stores the endpoints of the corresponding interval [C1(e),C2(e)] of Voronoi chains. Then there is a unique member C(e) of the interval [C1(e),C2(e)] which, in the search tree, is a common ascendant of all the other members of the interval (i.e. hierarchically the highest chain to which e belongs). Let C´be any such other member; then the discrimination (deciding on which side of C' the query point P lies) of a point P against C' is preceded - in the binary search scheme - by the discrimination of P against C(e). The computing of the common predecessor for the interval of chains in a standard hierarchy is described in [5,7] .

We have preprocessed our subdivision as in [5,7,8,9] . Note that the discrimination of a point against a chain C is effected in time O(log(n)) since each chain is the header of an adjacency list which contains the edges assigned to C. There are at most log(n) such discriminations since chains are organized in a balanced binary tree. So, the entire point location can be achieved in $O(\log^2(n))$ time (see [5,7,8,9] for details). Apparently the technique can be used for the location of O(n) points stored one point per processor in the same planar subdivision in $O(\log^2(n))$ time in parallel [7].

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Now, we can describe the merge step of our algorithm. Infinite rays of the contour can be computed as bisectors of tangent-lines of H(P) and H(Q) ([4,1,3]). We assign a processor to each edge of Vor(P) and Vor(Q). Let e=AB be an edge of Vor(P). Using the above technique we can locate A and B into Voronoi cells Vor(A1,Q) and Vor(B1,Q) of Vor(Q), respectively. Infinite rays can just compare their slopes with the slopes of infinite rays of contour. Then A and B compute their distances to the points A1 and B1 which are respectively their nearest points in P. In [1] it is claimed that there are three possible cases:

(1) both A and B are closer to P than to Q. Then AB remains a Voronoi edge [1];

(2) both A and B are closer to Q than to P. Then AB can be discarded from the Voronoi diagram [1] ;

(3) A is closer to P than to Q, but B is closer to Q than to P (the remaining case is discussed in a similar way). Then AB intersect the contour of the Voronoi diagram. The point B is discarded from Vor(S). Suppose AB is a bisector of points P1 and P2 in P. Then a new Voronoi vertex R is obtained as described below.

The set of all the new vertices R found in (3) can be sorted by their y-coordinates [2,6] and every edge defined by two neighbouring vertices in the sorted order is a Voronoi edge. The points that an edge UV is bisector of are determined by the common points associated with both U and V. For each new edge the endpoints of corresponding interval of chains it belongs to are just labels (ranks) of points the edge is bisector of. All the contours and edges of Vor(Q) that are included in Vor(S) increase their labels by $|P|$ (the number of points from P).

The algorithm is complete and works in $O(\log^3(n))$ time with $O(n \log(n))$ processors.

We are now going back to (3). Let the marked edges in Vor(P) or Vor(Q) be the edges for which one end is closer to P and the other end is closer to Q. A subtechnique for computing new Voronoi vertices, i.e. the points where marked edges intersect the contour presented in [1] and the next ina-

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Let P' and Q' denote the sets of marked points in P and Q respectively, i.e. the points in P and Q whose cells in Vor(Q) intersect the contour (the boundaries of these cells contain at least one marked edge). In [1] it is supposed that the marked points are sorted according to the order in which their cells intersect the contour. However, computing such an order is not a trivial problem, since this order is not necessarily the same as the order obtained by sorting according to the y-coordinates of points. Moreover, some points can appear twice in the sorted list of cells which intersect the contour. In the example in Fig. 1 corresponding orders are P'=(4,1,3,5), Q'=(9,7,6,9). Each marked edge determines a pair of neighbouring marked points in the sorted list P' (or Q').

We give the correct technique for ordering the marked points into a sorted list P'. A neighbouring pair of marked points is determined by a marked edge. Choose the endpoints of a neighbouring pair (of marked points) as the first and the last point in P'(and discard the pair from considerations in steps below). We have to compute the rank of each point in P'. Let the distance between two points in P' be the difference between their ranks in P'. For simplicity, we shall assume $n=2^k$ although the method can be easily modified to run for arbitrary n. In log(n) steps we proceed as follows:

For i=1 to i=log(n)-1 we can compute all the pairs of marked points that are at distance 2^i from each other. One such pair EF is determined by a point G such that G is at distance 2^{i-1} from both E and F.

Then, for i=log(n)-1 down to i=1 the points with ranks $m*2^i$ ($1 \leq i<n/2$), m odd) are found since for each of these points the ranks of two points which are at distance 2^i from the point are already computed.

For example let n=8 we are given circuit edges EF, CD, GH, DE, IJ, FG, HI where the ranks of C and J are 1 and 8 respectively. For i=1 we generate CE, EG, GI, DF, FH, HJ and for i=2 we get CG, EI, DH, FJ. Then, for i=2 G has rank 5 and F has rank 4. For i=1 we compute the ranks of D, E, H, I to be 2,3,6,7, respectively.

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Our method requires O (log(n)) steps and O (n*log(n)) space on a CREW PRAM.

We present a correct method for finding new Voronoi vertices. Choose the marked edge e determined by two median points A and B from P´. Let U and V be the endpoints of e such that U is closer to P than to Q. By assigning one processor to each point T´ from Q´ it is possible to determine in constant time for each T´, the intersection T between e and the bisector of AT´, and the distance of T to U. Then a point R´ from Q´ is chosen so that the considered distance RU is the smallest (we shall consider only the intersection points that lie between U and V). R is a new Voronoi vertex. It is easy to see that marked edges from the sorted list P´above (respectively, below) e can only interact with points above (below) R´ in the sorted list Q´. This partitioning is iterated log(n) times, after which all the contour edges can be computed.

3. PARALLEL CONVEX HULL IN THE PLANE

In [1] and [3] two similar convex hull algorithms which run in O (log(n)) time on a CREW PRAM with O (n) processors are described. We shall follow the algorithm in [1] and show that their last step is superfluous because they gave too weak a lemma.

Their algorithm divides the sorted list of n points from the given set S into $n^{1/2}$ sets of $n^{1/2}$ points, recursively the upper convex chain of each set computes, and then merges all these chains together in O (log(n)) time (similarly for the lower convex hull chains). The merge step begins by assigning a processor for each pair of $n^{(1/2)}$ upper chains and finding the "upper tangent line segments" of them. Consider the graph G whose vertices are the points of S and whose O (n) edges are either in the upper chains or among the just-computed upper tangent segments. After sorting the edges by the x-coordinate of their first components, all the edges with the same x-coordinate in the first component form a contiguous block. Similarly we can sort all the edges by the x-coordinate of their second component (this step is not mentioned in[1]). For each point $v \in S$, if E(v) denotes the edges incident to v, assign |E(v)| processors to v, and in O (log(n)) parallel time, sort E(v) by the slope of the

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edges. Notice that if v is a vertex on the upper chain of H(S), but not one of the two extremal vertices, then among the edges adjacent to v there are two cyclically adjacent edges which form a reflex angle with the interior pointing upward. Delete all the vertices (except the extremal vertices) which do not have two such incident edges, delete all but these two edges from those which do, and delete all the edges from the leftmost (respectively rightmost) extremal vertex except that with maximal (respectively minimal) slope. Denoting the trimmed graph by G´ in [1] the following lemma is stated.

LEMMA. Graph G´consists of one or more vertex chains of edges. One of these chains connects the extremal vertices, and this is the upper chain for H(S).

However, this assertion is too weak. We can prove a stronger result:

LEMMA. Graph G´is exactly the upper convex chain of S.

PROOF. It is obvious that all the vertices of G´ are vertices of the upper convex chain of S and for each vertex of G´ there are exactly two edges from G´ incident to the vertex (except the rightmost and leftmost vertices which are incident to only one edge). Therefore, no edge can be discarded from G´in order to obtain the upper convex hull.

Thus, the last step described in [1] is superfluous.

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REZIME

PARALELNO IZRAČUNAVANJE DIJAGRAMA VORONOJA

Rad prikazuje geometrijske algoritme za model paralelnog kompjutera u kome je dozvoljeno čitanje iz iste lokacije, ali ne i upis u isti registar. Prikazan je novi algoritam za paralelnu konstrukciju Voronoi dijagrama i pokazano da je raniji algoritam dat u [1] netačan. Takodje, algoritam za izračunavanje konveksnog omotača dat u [1] je napravljen efikasnijim izbacivanjem jednog suvišnog koraka.

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к-<2>-ПОЛУСЕТИ

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РЕЗЮМЕ

В работе определяются и исследуются к-<2>-полусети; $k \in N \setminus \{1,2\}$. к-<2>-полусети являются одним из обобщений к--полусетей, введеных автором в [1] как одно обобщение к-сетей [2 - 3]. В работе построени и примери конечных к-<2>-полусетей являющиеся аналогами аффинных пространств Спернера.

☆

Определение 1. Пусть $\mathfrak{T}$ непустое множество, элементы которого назовем *точки* и пусть $\mathfrak{L}$ непустое множество некоторых непустых подмножеств множества $\mathfrak{T}$, элементы которого назовем *линии*. Пусть, далее, множества $L_1,\ldots,L_k$, $k \in N \setminus \{1,2\}$, разбивают множество $\mathfrak{L}$. Объект $(\mathfrak{T},\{L_1,\ldots,L_k\})$ назовем к-<2>-*полусетью* тогда и только тогда, когда выполняются следующие условия:

M1 $(\forall \ell \in \mathfrak{L})(\forall \ell' \in \mathfrak{L})(\ell \in L_i \wedge \ell' \in L_j \wedge i \neq j \Rightarrow$
$\Rightarrow |\ell \cap \ell'| \leqslant 2)$[1]; и

[1] Две линии различных классов пересекаются не больше, чем в двух точках.

AMS Mathematics Subject Classification (1980): 20N05.

Key words and phrases: k-nets, k-seminets, A_t-3-quasigroups, A_t^m--3-quasigroups.

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M2 $(\forall A \in \mathfrak{T})(\exists! \ell_1 \in L_1) \ldots (\exists! \ell_k \in L_k)(A \in \ell_1 \wedge \ldots \wedge A \in \ell_k)$[1].

Объект $(\mathfrak{T}, \mathfrak{L})$, где $\mathfrak{L} = \sum_{i=1}^{k} L_i$, назовем *носителем* к-<2>-полусети $(\mathfrak{T}, \{L_1, \ldots, L_k\})$.

Если в определении 1 вместо M1 берется

M1′ $(\forall \ell \in \mathfrak{L})(\forall \ell' \in \mathfrak{L})(\ell \in L_i \wedge \ell' \in L_j \wedge i \neq j \Rightarrow \Rightarrow |\ell \cap \ell'| \leqslant 1)$;

то $(\mathfrak{T}, \{L_1, \ldots, L_k\})$ станет к-полусетью [1]. Отсюда получаем, что имеет место следующее утверждение:

Утверждение 1. Если объект $(\mathfrak{T}, \{L_1, \ldots, L_k\})$ является к-полусетью (к-сетью [2]), то $(\mathfrak{T}, \{L_1, \ldots, L_k\})$ является к-<2>-полусетью.

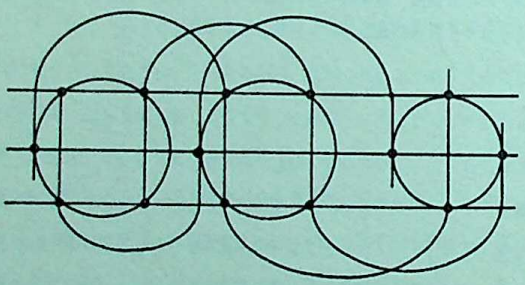

Рис. 1_1

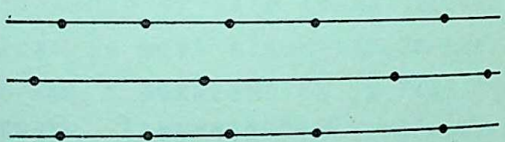

Рис. 1_2

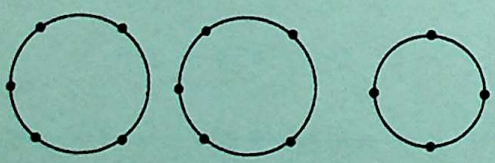

Рис. 1_3

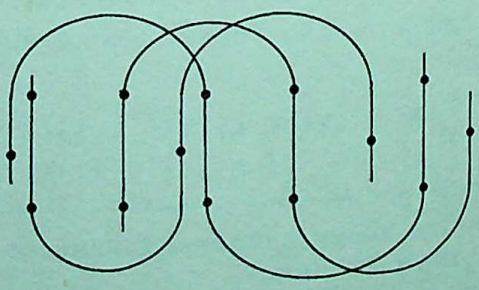

Рис. 1_4

1) Каждая точка из $\mathfrak{T}$ находится в одной и только в одной линии каждого класса L_i, $i \in \{1, \ldots, k\}$.

2) к-полусети, описанные автором в [1], являются одним обобщением к-сетей [2 - 3].

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На рис. 1_1 - 1_4 изображена 3-<2>-полусеть[1] удовлетворяющая условию: существуют линии $\ell \in L_i$, $\ell' \in L_j$, $i \neq j$ такие что $|\ell \cap \ell'| = 2$. Таким образом, имеет место следующее утверждение:

Утверждение 2. Существуют к-<2>-полусети, не являющиеся к-полусетями.

Если в определении 1 вместо $к \in N \setminus \{1,2\}$ берется $к \in N \setminus \{1,\ldots,n\}$, $n \in N \setminus \{1\}$, а вместо М1 берется

$$M1'' \quad (\forall \ell_1 \in \mathfrak{L}) \ldots (\forall \ell_n \in \mathfrak{L})(\ell_1 \in L_{i_1} \wedge \ldots \wedge \ell_n \in L_{i_n} \wedge \wedge |\{i_1,\ldots,i_n\}| = n \Rightarrow |\ell_1 \cap \ldots \cap \ell_n| = 1);$$

то $(\mathfrak{k},\{L_1,\ldots,L_k\})$ станет <к,n>-сетью[2]. Для $n = 2$ речь идет о к-сетях. В <к,n>-сетях имеет место: если $\ell_1 \in L_i \wedge \ell_2 \in L_j \wedge i \neq j$, то $|\ell_1 \cap \ell_2| = |Q^{n-2}|$. Отсюда получаем: если $n \geqslant 3$ и $|Q| = |L_i| > 2$, то <к,n>-сеть не является к-<2>-полусетью. Притом, в <к,n>-сетях[3] справедливо:

$$M\bar{1} \quad (\forall \ell_1 \in \mathfrak{L}) \ldots (\forall \ell_к \in \mathfrak{L})(\ell_1 \in L_1 \wedge \ldots \wedge \ell_к \in L_к \Rightarrow \Rightarrow |\ell_1 \cap \cdots \cap \ell_к| \leqslant 1).$$

Очевидно, что условие $M\bar{1}$ справедливо и в к-полусетях. В самом деле, если в определении 1 вместо М1 берется $M\bar{1}$, то получается обобщение: к-полусетей, <к,n>-полусетей[3], сетей Тайлора из [13], сетей Хавела из [14 - 15][4].

На рис. 2_1 - 2_4 изображена 3-<2>-полусеть[5] $(\mathfrak{k},\{L_1, L_2, L_3\})$ удовлетворяющая условию: существуют линии $\ell_1 \in L_1$, $\ell_2 \in L_2$ и $\ell_3 \in L_3$ такие, что имеет место $|\ell_1 \cap \ell_2 \cap \ell_3| = 2$. Отсюда получаем, что справедливо:

---

1) На рис. 1_1 изображен носитель, а на рис 1_2 - 1_4 изображены классы.

2) <4,3>-сети введены в [4], <n+1,n>-сети в [6], а <к,n>-сети в [7]. <к,n>-сети рассматриваны, например, и в работах: [5,8-11].

3) и <к,n>-полусетях. <к,n>-полусети введены в [12] как обобщение к-полусетей и <к,n>-сетей.

4) Речь идет о частном случае сетей рассматриваемых В. Хавелом в [16]. Некоторым образом условие $M\bar{1}$ рассматривается и Р. Галичем в [17 - 18].

5) На рис. 2_1 изображен носитель, а на рис. 2_2 - 2_4 изображены классы.

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Утверждение 3. Существуют к-<2>-полусети не удовлетворяющие условию M1̄.

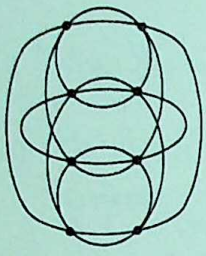

Рис. 2_1

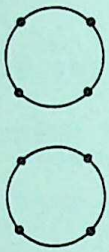

Рис. 2_2

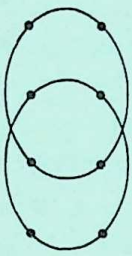

Рис. 2_3

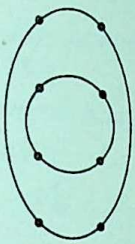

Рис. 2_4

Следующее утверждение является следствием аксиомы M2:

Утверждение 4. Пусть $(\mathfrak{T},\{L_1,\ldots,L_k\})$ к-<2>-полусеть. Тогда имеет место:

$$(\forall \ell \in \mathfrak{L})(\forall \ell' \in \mathfrak{L})(\ell \in L_i \wedge \ell' \in L_j \wedge i = j \wedge \ell \neq \ell' \Rightarrow \Rightarrow |\ell \cap \ell'| = 0)^{1)}.$$

к-<2>-полусети изображены, в том же порядке, на рис. 1_1 - 1_4 и рис. 2_1 - 2_4 удовлетворяют следующему условию:

M3 $(\forall \ell \in \mathfrak{L})(\forall \ell' \in \mathfrak{L})(\ell \in L_i \wedge \ell' \in L_j \wedge i \neq j \Rightarrow \Rightarrow |\ell \cap \ell'| \geqslant 1)$.

На рис. 3_1 - 3_3 изображены классы 3-<2>-полусети не удовлетворяющей условию M3.

Учитывая M1̄ и M2, находим, что имеет место следующее утверждение:

Утверждение 5. Если в к-<2>-полусети $(\mathfrak{T},\{L_1,\ldots,L_k\})$ существуют классы $L_i, L_j, i \neq j$, такие, что для каждых $\ell \in L_i$ и

1) Различные линии принадлежащие одному и тому же классу не пересекаются.

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$\ell' \in L_j$ имеет место $|\ell \cap \ell'| \leqslant 1$, тогда в $(\mathfrak{T}, \{L_1, \ldots, L_к\})$ имеет место $M\bar{1}$.

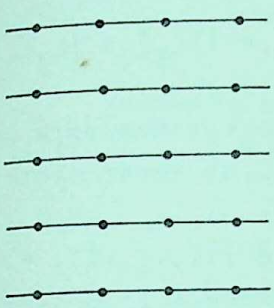

Рис. 3_1

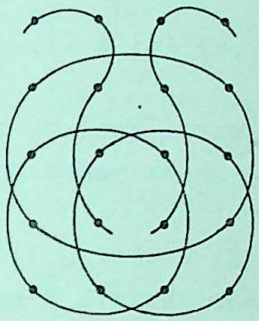

Рис. 3_2

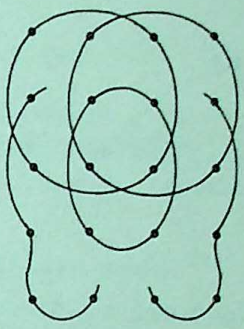

Рис. 3_3

Обратное не имеет место. Именно, в 4-<2>-полусети изображеной на рис. 2_1 - 2_5 выполняются следующие условия:

а) $M\bar{1}$; и

б) пересечение каждых двух линии принадлежащих различным классам является двухэлементным множеством.

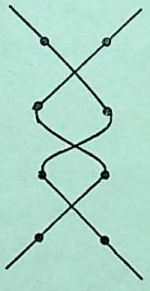

Рис. 2_5

Пусть $(\mathfrak{T}, \{L_1, \ldots, L_к\})$ конечная к-<2>-полусеть; $|\mathfrak{T}| \in N$. Пусть, далее, имеет место

$$(\forall \ell \in \mathfrak{L})|\ell| = m \in N \setminus \{1,2\}.$$

Тогда, ввиду M2 (и утверждения 4), имеет место: $|L_i| \cdot m = |L_j| \cdot m = |\mathfrak{T}|$ для любых $i,j \in \{1,\ldots,к\}$. Отсюда получаем, что справедливо:

Утверждение 6. Если в конечной к-<2>-полусети $(\mathfrak{T}, \{L_1, \ldots, L_к\})$ имеет место

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M4 $(\forall \ell \in \mathfrak{L})(\forall \ell' \in \mathfrak{L})|\ell| = |\ell'|$,

тогда в $(\mathfrak{T}, \{L_1, \ldots, L_\kappa\})$ имеет место:

M5 $(\forall i \in \{1, \ldots, \kappa\})(\forall j \in \{1, \ldots, \kappa\})|L_i| = |L_j|$.

Обратное не имеет место. Именно, в 3-<2>-полусети изображеной на рис. 1_1 - 1_4 имеет место M5 и не имеет место M4.

Числа $\max\{|\ell| \mid \ell \in \mathfrak{L}\}$ и $\max\{|L_i| \mid i \in \{1, \ldots, \kappa\}\}$, в том же порядке, позволим себе назвать T-порядок и L-порядок к-<2>-полусети $(\mathfrak{T}, \{L_1, \ldots, L_\kappa\})$[1].

Теорема 7. Пусть m = T-порядок и q = L-порядок конечной к-<2>-полусети. Тогда имеет место:

$$q \geqslant \begin{cases} m/2, & m = 2s \\ (m+1)/2, & m = 2s-1 \end{cases}, \quad s \in N \setminus \{1\}.$$

Доказательство. Пусть $(\mathfrak{T}, \{L_1, \ldots, L_\kappa\})$ конечная к-<2>--полусеть. Пусть, далее, $\ell \in \mathfrak{L}$ такая, что T-порядок = $|\ell| = m$, а класс L_i такой, что L-порядок = $|L_i| = q$. Притом, пусть класс L_j такой, что $\ell \notin L_j$. Отсюда получаем, что

(а) $|L_j| \leqslant |L_i|$ (= L-порядок).

Справедливо:

(б) каждая точка $A \in \ell$ находится в одной и только в одной линии класса L_j (M2); и

(ц) $|\ell \cap \ell'| \leqslant 2$ для каждой $\ell' \in L_j$ (M1).

Учитывая (б) и (ц), находим, что имеет место:

[1] Таким же образом в [1] определени T-порядок и L-порядок к-полусетей.

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(д) $|\{\ell' | \ell' \in L_j \wedge \ell' \cap \ell \neq \phi\}| \geq \frac{m}{2}$ если $m = 2s$[1]

и

(е) $|\{\ell' | \ell' \in L_j \wedge \ell' \cap \ell \neq \phi\}| \geq \frac{m+1}{2}$ если $m = 2s - 1$[2].

Учитывая (а), (д) и (е), ввиду отношения $|L_j| \geqslant |\{\ell' | \ell' \in L_j \wedge \ell' \cap \ell \neq \phi\}|$[3], находим, что теорема доказана.

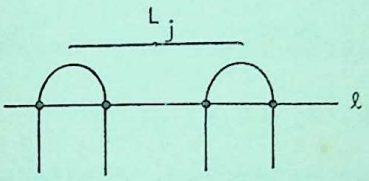

Рис. 4_1

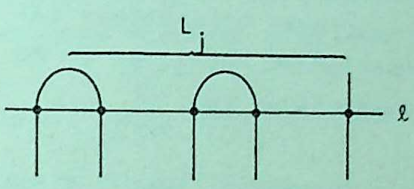

Рис. 4_2

В 4-<2>-полусети изображеной на рис. 2_1 - 2_5 имеет место: $q = m/2 = 4/2 = 2$. В 3-<2>-полусети изображеной на рис. 1_1 - 1_4 имеет место: $q = (m+1)/2 = (5+1)/2 = 3$. В 3-<2>-полусети изображеной на рис. 3_1 - 3_3 имеет место: $q = 5 > m/2 = 4/2 = 2$[4].

Теорема 8. Пусть $(\mathfrak{T}, \{L_1, \ldots, L_к\})$ к-<2>-полусеть, а $\|$ бинарное отношение в $\mathfrak{L} = L_1 \cup \ldots \cup L_к$, определяемое следующим образом:

(е) $p \| q \overset{\text{деф}}{=} p \in L_i \wedge q \in L_j \wedge i = j.$

Тогда в объекте $(\mathfrak{T}, \mathfrak{L}, \|)$ имеет место:

---

1) Рис. 4_1.
2) Рис. 4_2.
3) М1.
4) В самом деле: $q > m$.

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A1 $(\forall \ell \in \mathfrak{L})(\forall \ell' \in \mathfrak{L})(\ell \neq \ell' \Rightarrow |\ell \cap \ell'| \leqslant 2)$;

A2 каждая точка $A \in \mathfrak{T}$ находится в к ($\in N \setminus \{1,2\}$) и только в к ($\in N \setminus \{1,2\}$) попарно различных элементов множества $\mathfrak{L}$;

A3 $\parallel$ является РСТ отношением в $\mathfrak{L}$; и

A4 $(\forall A \in \mathfrak{T})(\forall \ell \in \mathfrak{L})(\exists! \ell' \in \mathfrak{L})(\ell \parallel \ell' \wedge A \in \ell')$[1].

Доказательство. A1 является следствием аксиомы M1 и утверждения 4. A2 является следствием аксиомы M2. A3 является следствием определения (e) и факта, что $\{L_1,\ldots,L_к\}$ является разбиением множества $\mathfrak{L}$. A4 является следствием аксиомы M2 и определения (e).

Теорема 9. Пусть $\mathfrak{T}$ непустое множество, элементы которого назовем точками и пусть $\mathfrak{L}$ непустое множество некоторых непустых подмножеств множества $\mathfrak{T}$, элементы которого назовем линии. Пусть, далее, $\parallel$ бинарное отношение в $\mathfrak{L}$. Тогда, если в объекте $(\mathfrak{T},\mathfrak{L},\parallel)$ выполняются условия A1 - A4 (из теоремы 8), тогда объект $(\mathfrak{T},\{L_i|i \in I\})$, где $\{L_i|i \in I\} \overset{\text{деф}}{=} \mathfrak{L}/\parallel$, является к-<2>-полусетью.

Доказательство.

1) Ввиду A3, пусть $\{L_i|i \in I\} \overset{\text{деф}}{=} \mathfrak{L}/\parallel$. Отсюда, учитывая A4 и A2, находим, что имеет место:

1_1) $(\forall \ell \in \mathfrak{L})(\forall \ell' \in \mathfrak{L})(\ell \in L_i \wedge \ell' \in L_j \wedge i = j \wedge \ell' \neq \ell \Rightarrow |\ell \cap \ell'| = 0)$; и

1_2) $|\{L_i|i \in I\}| = к$, $к \in N \setminus \{1,2\}$.

2) Учитывая A1 и 1_1), находим, что в $(\mathfrak{T},\{L_i|i \in I\})$ имеет место M1.

3) Учитывая 1_2), A2 и A4, находим, что в $(\mathfrak{T},\{L_1,\ldots,L_к\})$ имеет место M2.

Теорема доказана.

---

[1] Аксиома евклидовой параллельности.

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Пусть $\mathfrak{T}$ непустое множество и пусть непустое множество $\mathfrak{L}$ множество некоторых непустых подмножеств множества $\mathfrak{T}$. Множество $\mathfrak{L}$ называется разбиением Хартманиса типа n, $n \in N$, множества $\mathfrak{T}$[1] тогда и только тогда, когда выполняются следующие условия:

H1 $(\forall A_1 \in \mathfrak{T}) \ldots (\forall A_n \in \mathfrak{T})(|\{A_1^n\}| = n \Rightarrow$
$\Rightarrow (\exists! \ell \in \mathfrak{L})(A_1 \in \ell \wedge \ldots \wedge A_n \in \ell)$; и

H2 $(\forall \ell \in \mathfrak{L})|\ell| \geq n$ [21][2].

Объект $(\mathfrak{T},\mathfrak{L})$ позволим себе назвать nH-геометрией. Притом, элементы множества $\mathfrak{T}$ назовем точками, а элементы множества $\mathfrak{L}$ линиями.[3]

Пусть $(\mathfrak{T},\mathfrak{L})$ nH-геометрия. Тогда имеет место:

H1′ $(\forall \ell \in \mathfrak{L})(\forall \ell' \in \mathfrak{L})(\ell \neq \ell' \Rightarrow |\ell \cap \ell'| \leqslant n-1)$; и

H1′′ $(\forall A \in \mathfrak{T})(\exists \ell \in \mathfrak{L})A \in \ell$.

Притом, существуют объекты $(\mathfrak{T},\mathfrak{L})$ удовлетворяющие условиям H1′, H1′′, H2 и не удовлетворяющие условию H1. Объект $(\mathfrak{T},\mathfrak{L})$ называется PnH-геометрия тогда и только тогда, когда выполняются условия H1′, H1′′ и H2 [25][4].

Носители к-<2>-полусетей изображеных на рис. 1 - 3 являются PnH-геометриями и не являющиеся nH-геометриями. На рис. 5_1 - 5_4 изображена 3-<2>-полусеть имеющая носитель удовлетворяющий условиям H1′ и H1′′ (n = 3) и не удовлетворяющий условию H2. Объект $(\mathfrak{T},\mathfrak{L})$ удовлетворяющий условиям H1′ и H1′′, $n \in N \setminus \{1\}$, позволим себе назвать $L^{(n-1)}$-геометрией.

Носитель 7-<2>-полусети изображеной на рис. 2_1 - 2_8 является 3H-геометрией. Притом, имеет место:

---

1) Короче: nH-разбиением множества $\mathfrak{T}$.
2) Каждому nH-разбиению множества $\mathfrak{T}$ соотвествует (n+1)-арное отношение эквивалентности на множестве $\mathfrak{T}$, и обратно [22].
3) Каждой A_t-квазигруппе (каждому A_t-групоиде) соотвествует 2H-геометрия [23,24]. Подобно, каждой A_t-3-квазигруппе соотвествует 3H-геометрия [26].
4) Каждому частичному A_t-группоиде соответствует PnH-геометрия [25].

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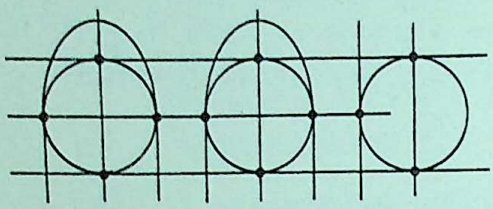

Рис. 5_1

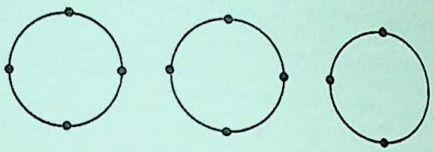

Рис. 5_2

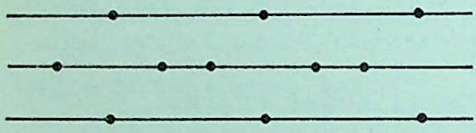

Рис. 5_3

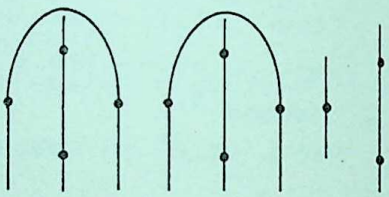

Рис. 5_4

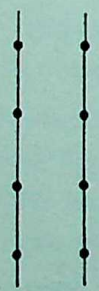

Рис. 2_6

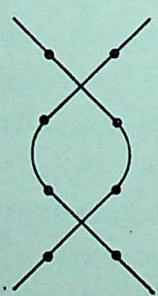

Рис. 2_7

Рис. 2_8

НЗ $(\forall \ell \in \mathfrak{L})(\forall \ell' \in \mathfrak{L})|\ell| = |\ell'|$ [1].

[1] В самом деле, носитель этой 7-<2>-полусети является 3-(8,4,1)-схемой [28]; $(\forall \ell \in \mathfrak{L})|\ell| = 4$, $|\mathfrak{T}| = 8$.

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Теорема 10. Если в конечной к-<2>-полусети $(\mathfrak{T},\{L_1,\ldots,L_к\})$, $|\mathfrak{T}| = t$, имеет место

$$Н\bar{3} \quad (\forall \ell \in \mathfrak{L})|\ell| = m \in N \setminus \{1,2\},$$

то

$$к \leqslant (t-1)(t-2)/2\cdot\binom{m-1}{2}.$$

Притом, имеет место:

$$Н1 \Leftrightarrow (t-1)(t-2)/2\cdot\binom{m-1}{2}.$$

Доказательство. Пусть A произвольная точка из $\mathfrak{T}$. В $\mathfrak{T}$, $|\mathfrak{T}| = t$, существует $(t-1)(t-2)/2$ подмножеств $\{B,C\}$ таких, что $|\{A,B,C\}| = 3$. Ввиду А1, находим, что имеет место: существует одна и только одна линия $\ell \in \mathfrak{L}$ такая, что $\{A,B,C\} \subseteq \ell$. Притом, существует $m-3$ точек $D \in \mathfrak{T}$ таких, что $D \in \ell$ и $D \notin \{A,B,C\}$; $Н\bar{3}$. Таким образом, через произвольную точку $A \in \mathfrak{T}$ проходит не больше чем $(t-1)(t-2)/2\cdot\binom{m-1}{2}$ линий. В самом деле, если через каждую точку A проходит $(t-1)(t-2)/2\cdot\binom{m-1}{2}$ линий, то в $(\mathfrak{T},\{L_1,\ldots,L_к\})$ справедливо Н1, и обратно. Наконец, учитывая М2, находим, что в к-<2>-полусетях удовлетворяющих условию $Н\bar{3}$ имеет место отношение: $к \leqslant (t-1)(t-2)/2\cdot\binom{m-1}{2}$.

Примечание 1. Ввиду теоремы 8 и теоремы 9 имеет смысл объект $(\mathfrak{T},\mathfrak{L},\|)$, удовлетворяющий условиям А1 - А4, считать к-<2>-полусетью; $к = |\mathfrak{L}/\||$. Учитывая теорему 10, находим, что конечные к-<2>-полусети удовлетворяющие условиям Н1 и $Н\bar{3}$ можно описать следующими условиями:

SA1 через каждую тройку попарно различных точек проходит одна и только одна линия[1];

SA2 $(\forall \ell \in \mathfrak{L})|\ell| = m \in N \setminus \{1,2\}$[2];

---

[1] Н1 для $n = 3$.

[2] $Н\bar{3}$.

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SA3 отношение $\parallel$ является PCT отношением на множестве $\mathfrak{L}$[1]; и

SA4 $(\forall \ell \in \mathfrak{L})(\forall A \in \mathfrak{T})(\exists! \ell' \in \mathfrak{L})(\ell \parallel \ell' \wedge A \in \ell')$[2].

Отметим: если в системе SA1 - SA4 вместо SA1 берется

SA$\bar{1}$ через каждую двойку различных точек проходит одна и только одна линия;

то $(\mathfrak{T},\mathfrak{L},\parallel)$ станет аффинным пространством Спернера [19]. Притом, если $q = m$, где $q = |L_i|$, $i \in \{1,\ldots,\kappa\}$, то $(\mathfrak{T},\mathfrak{L},\parallel)$ является аффинной плоскостью.

На рис. 6_1 - 6_{28} изображена 28-<2>-полусеть удовлетворяющая условиями H1 и H$\bar{3}$; $|\mathfrak{T}| = 9$, $m = q = 3$. На рис. 7_1 - 7_{35} изображена 35-<2>-полусеть удовлетворяющая условиями H1 и H$\bar{3}$; $|\mathfrak{T}| = 16$, $m = q = 4$. Ее носитель построен на основании теоремы 2 и утверждения 6 из, в том же порядке, работ [27] и [26]. Отметим: если вместо классов изображеных на рис. 7_3 - 7_8 берется классы тех же линий изображеных на рис. 8_1 - 8_6, то получаем новую 35-<2>-полусеть. Притом, $\mathfrak{T}$ в отношении на множество классов изображеных на рис. 7_1 - 7_5 является 5-сетью порядка 4, т.е. аффинной плоскостью порядка 4.

Примечание 2. Условие

$$(\forall \ell \in \mathfrak{L})(\forall \ell' \in \mathfrak{L})(\ell \in L_i \wedge \ell' \in L_j \wedge i \neq j \Rightarrow$$
$$\Rightarrow |\ell \cap \ell'| \leqslant n-1)^3$$

для $n = 3$ превращается в условие под M1, а для $n = 2$ в условие под M1'.

---

1) A3.

2) A4.

3) $n \in N$.

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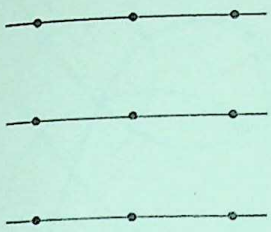


Рис. 6$_1$

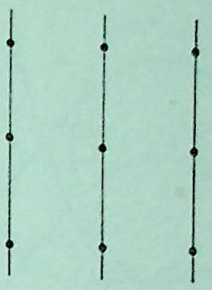


Рис. 6$_2$

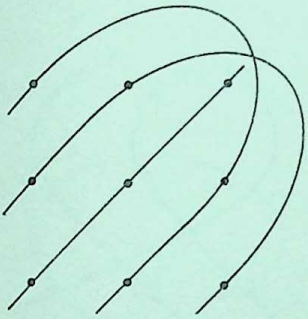


Рис. 6$_3$

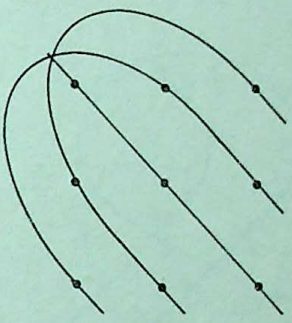


Рис. 6$_4$

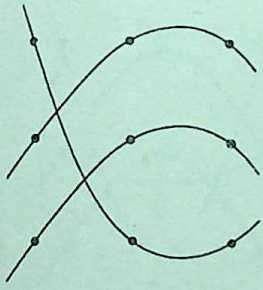
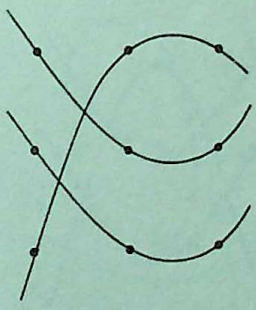


Рис. 6$_5$



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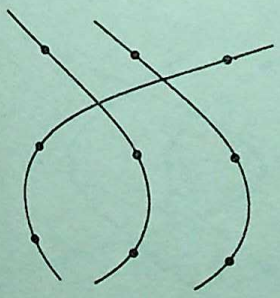

Рис. 6$_7$

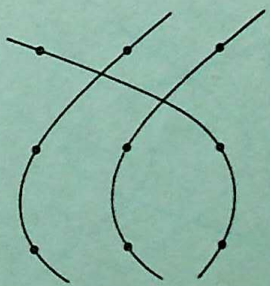

Рис. 6$_8$

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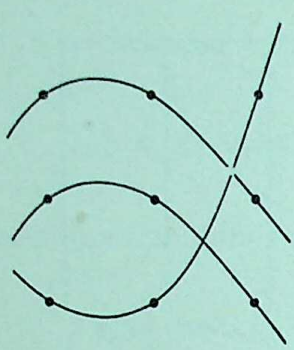


Рис. 6_9

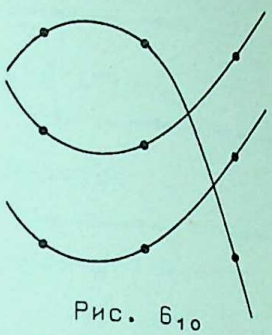


Рис. 6_{10}

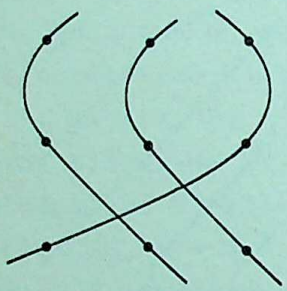


Рис. 6_{11}

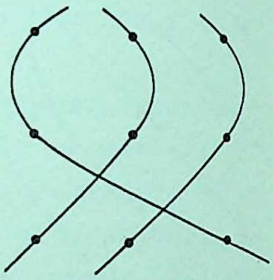
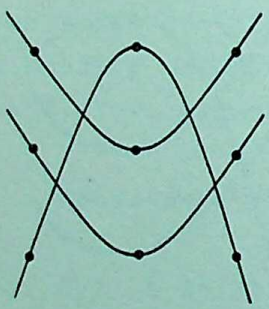


Рис. 6_{12}



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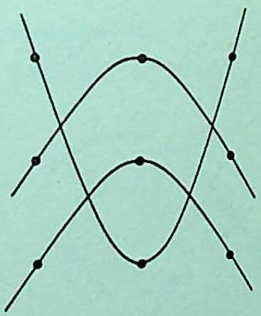

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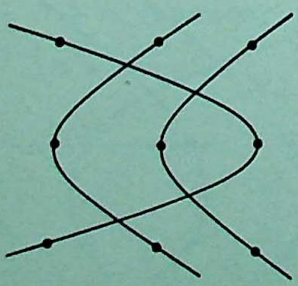

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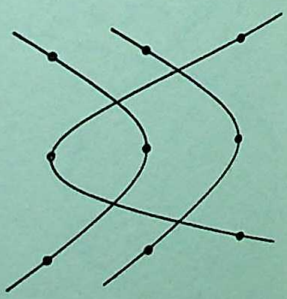

Рис. 6_{16}

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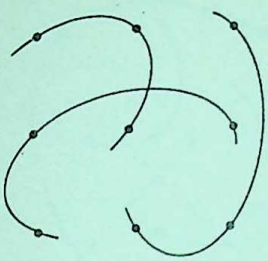

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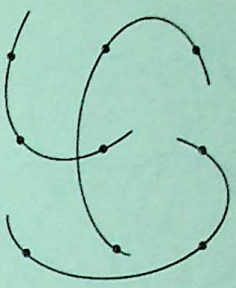

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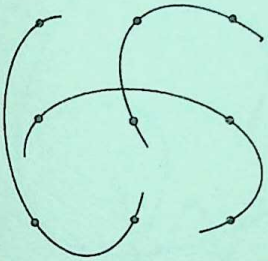

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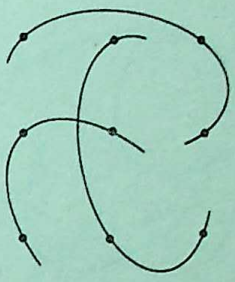

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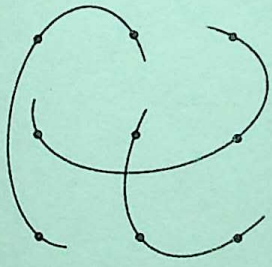

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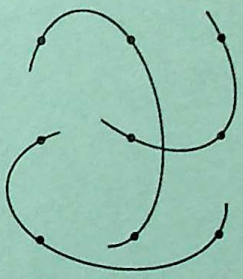

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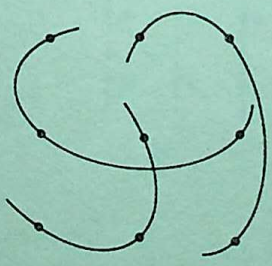

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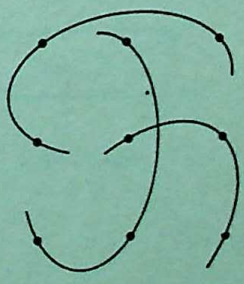

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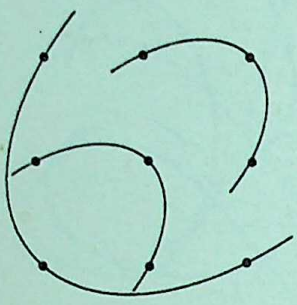

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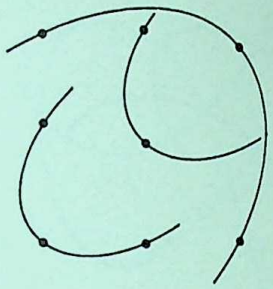

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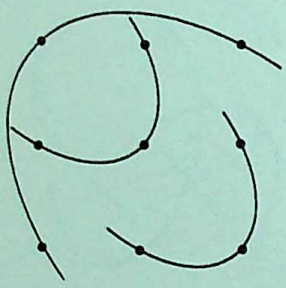

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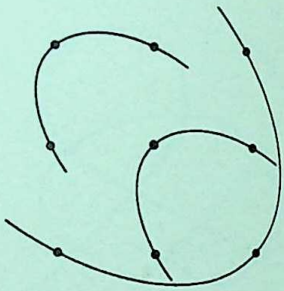

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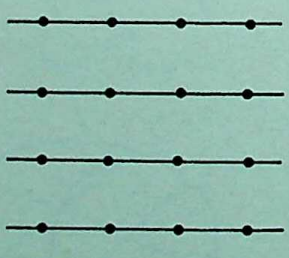

Рис. 7_1

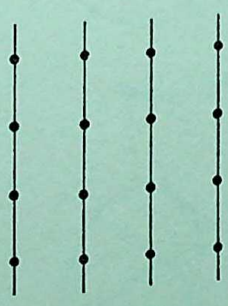

Рис. 7_2

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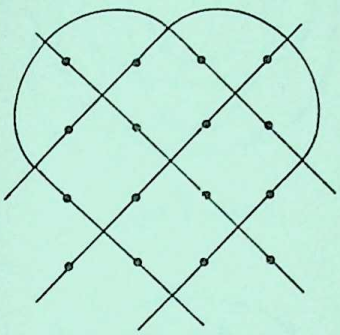
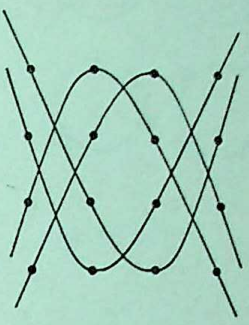


Рис. 7з



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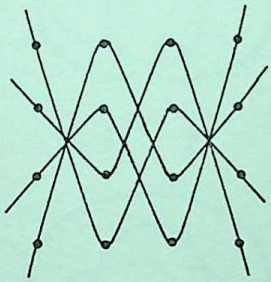

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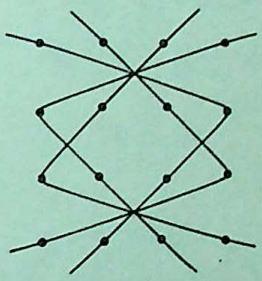

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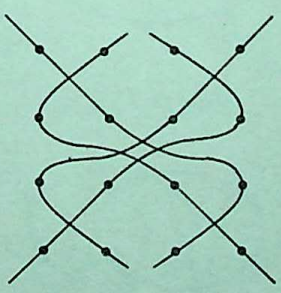

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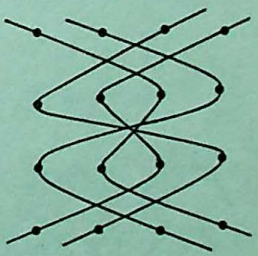

Рис. 7₈

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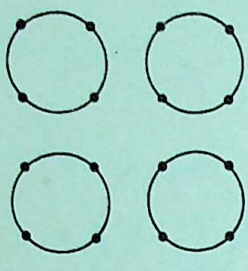

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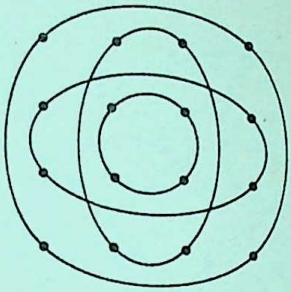

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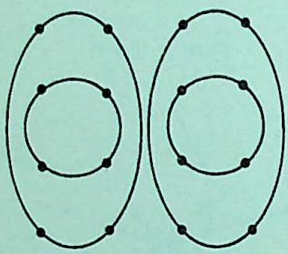

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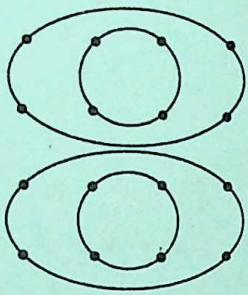

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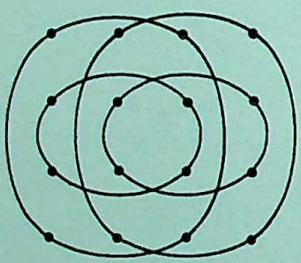

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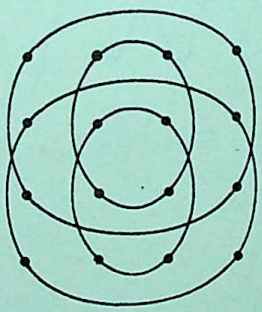

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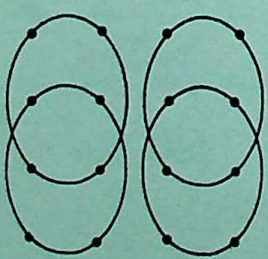

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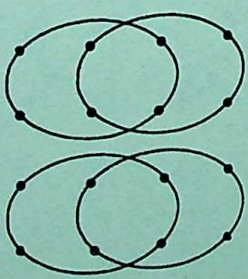

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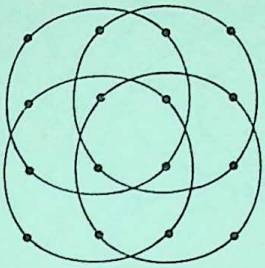

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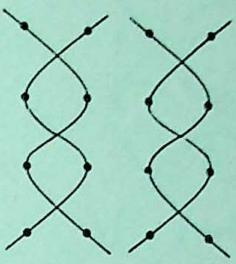

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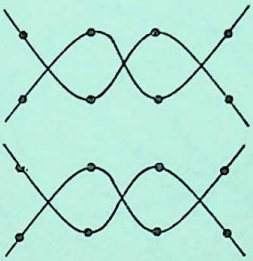

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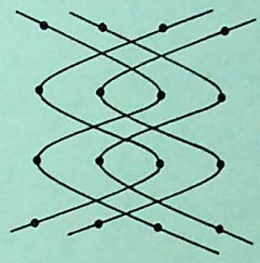

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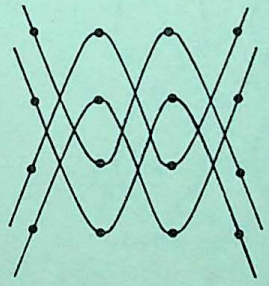

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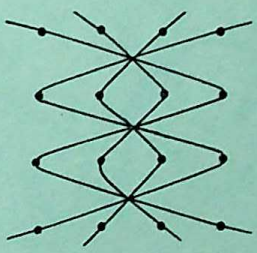

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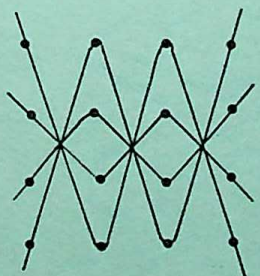

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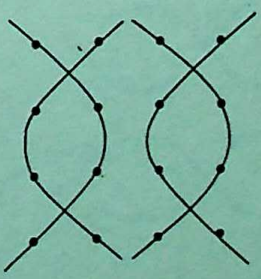

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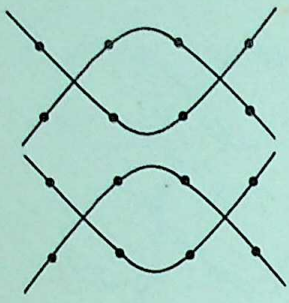

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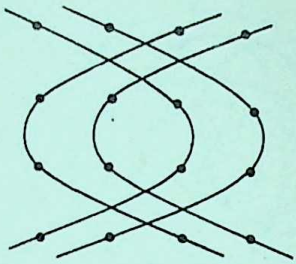

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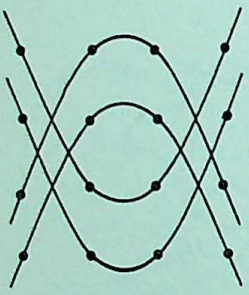

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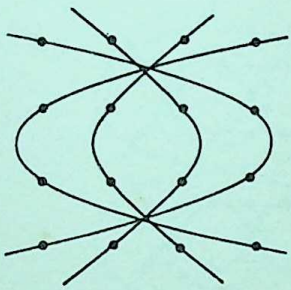

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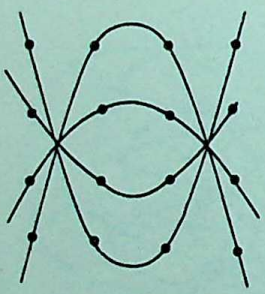

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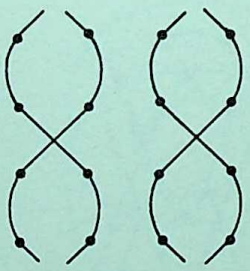

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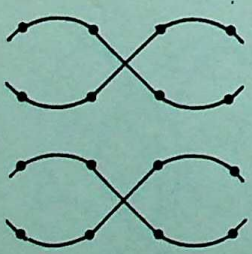

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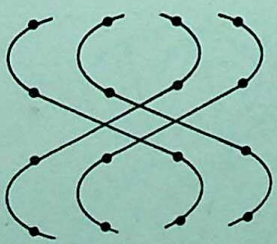

Рис. 7_{32}

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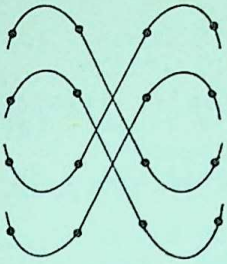

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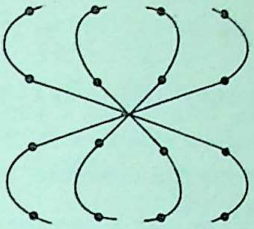

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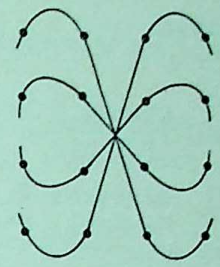

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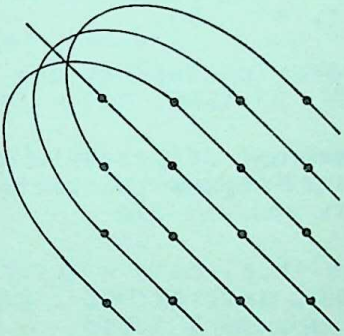

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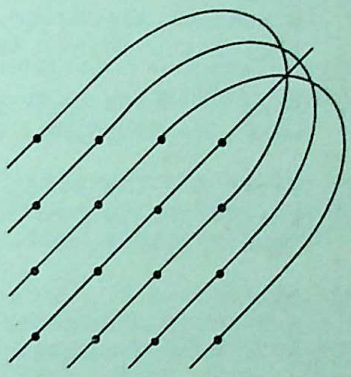

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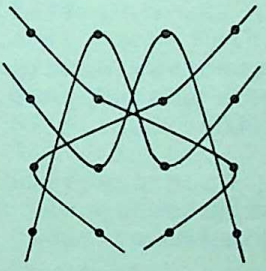

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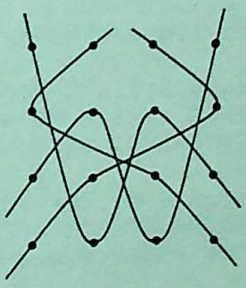

Рис. 8_4

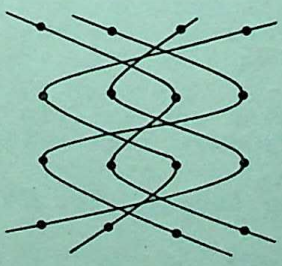

Рис. 8_5

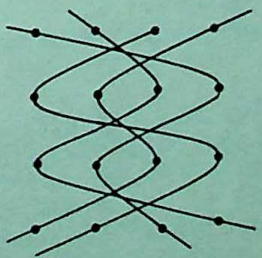

Рис. 8_6

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РЕЗИМЕ

К-<2>-ПОЛУРЕШЕТКЕ

U radu se uvode i razmatraju k-<2>-polurešetke kao jedno uopštenje k-polurešetaka [1], odnosno k-rešetaka [2 - 3]. Izmedju ostalog, u radu se opisuje i nekoliko k-<2>-polurešetaka sa najvećim mogućim $k \in N \setminus \{1,2\}$ pri datim uslovima.

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